Theoretical Foundations II: Structure of Rough Sets

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Borders

- In case of "classical sets", there is no ambiguity, whether an element belongs to a set or not: either $a \in X$ or $a \notin X$
- The **border** between the set X and its complement X^c is edge-sharp: no element can sit in the border
- In case of "rough sets", the situation is different. The **border** $B(X) = X^{\blacktriangle} \setminus X^{\blacktriangledown}$ is the area of uncertainty.
- If $a \in B(X)$, then both in X and outside X there are elements to which a is R-related – here R is the relation representing our knowledge
- Based on this difference, the structure of rough sets is quite different from the structure of the "classical" sets

Definition of rough sets

Let R be a relation representing our knowledge and let rough approximations be formed by this knowledge.

- A **rough equivalence relation:** two sets X and Y are **roughly equivalent**, denoted by $X \equiv Y$, if $X^{\blacktriangledown} = Y^{\blacktriangledown}$ and $X^{\blacktriangle} = Y^{\blacktriangle}$
- This means that $X \equiv Y \iff$ the set X and Y look exactly similar in view of the knowledge R
- The equivalence classes $[X]_{\equiv} = \{Y \mid X \equiv Y\}$ of $\equiv$ are called **rough sets**
- This really is the original (1981) definition by Pawlak
- The relation $\equiv$ can be viewed as an **indiscernibility** relation, but between sets.

Example

Let $U = \{a, b, c\}$ and let E be an equivalence on U such that its equivalence classes are $\{a, b\}$ and $\{c\}$.

X	$X^{\blacktriangledown}$	$X^{\blacktriangle}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\{a, b\}$
$\{c\}$	$\{c\}$	$\{c\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{c\}$	U
$\{b, c\}$	$\{c\}$	U
U	U	U

Rough sets are:

(i): $\{\emptyset\}$, (ii): $\{\{a\}, \{b\}\}$, (iii): $\{\{c\}\}$,

(iv): $\{\{a, b\}\}$, (v): $\{\{a, c\}, \{b, c\}\}$, (vi): $\{U\}$.

Ordered set of rough sets

- Considering rough sets as equivalence classes of sets is not very practical.
- On the other hand, each rough set $[X]_\equiv$ is uniquely determined by the approximation pair $(X^\blacktriangledown, X^\blacktriangle)$.
- We use approximation pairs instead of equivalence classes
- The set of all **rough sets** is $RS = \{(X^\blacktriangledown, X^\blacktriangle) \mid X \subseteq U\}$
- We obtain an ordered set $\mathcal{RS} = (RS, \leq)$ by ordering RS by the **coordinatewise order**:

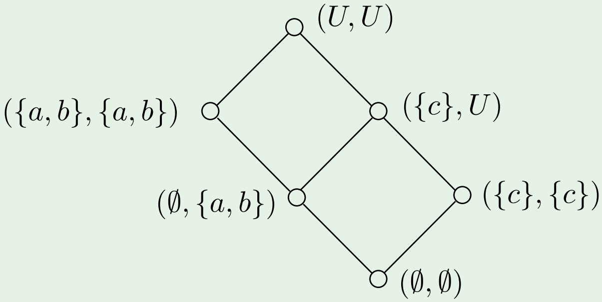
$$(X^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, Y^\blacktriangle) \iff X^\blacktriangledown \subseteq Y^\blacktriangledown \text{ and } X^\blacktriangle \subseteq Y^\blacktriangle$$

Example

In our previous example,

$$RS = \{(\emptyset,\emptyset),\ (\emptyset,\{a,b\}),\ (\{a,b\},\{a,b\}),\\ (\{c\},\{c\}),\ (\{c\},U),\ (U,U)\}$$

The ordered set $\mathcal{RS}$ has the following structure:



It seems to be isomorphic to $\mathbf{2}\times\mathbf{3}$

Structure of "classical sets"

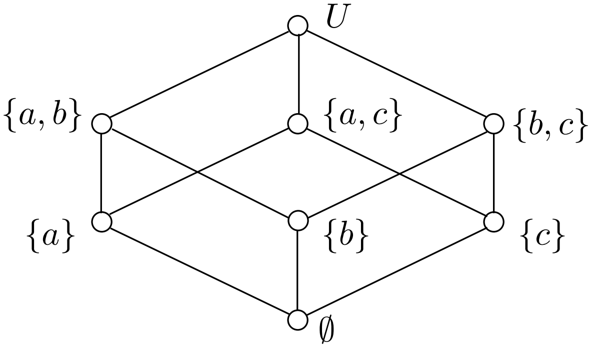
- The ordered set $(\wp(U), \subseteq)$ of all subsets of U is a complete lattice such that for all $\mathcal{H} \subseteq \wp(U)$:

$$\bigvee \mathcal{H} = \bigcup \mathcal{H} \text{ and } \bigwedge \mathcal{H} = \bigcap \mathcal{H}$$

- In particular, $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$
- $(\wp(U), \cup, \cap, {}^c, \emptyset, U)$ is a Boolean algebra, where $X^c = U \setminus X$ is the complement of X.
- It is known from the general lattice-theory that a Boolean lattice is atomistic if and only if it is completely distributive
- Atoms of $\wp(U)$ are the singletons $\{x\}$ for $x \in U$.
- $\wp(U) \cong \mathbf{2}^U$.

Structure of "classical sets"

Let $U = \{a, b, c\}$. The complete lattice $(\wp(U), \subseteq)$ is:



The structure of $\mathcal{RS}$ in case of an equivalence relation

- Let $\mathcal{RS}$ be determined by an equivalence relation E.
- The **cartesian product** $\wp(U) \times \wp(U) = \{(X, Y) \mid X, Y \subseteq U\}$ is a complete lattice such that for each subset $\{(X_i, Y_i)\}_{i \in I}$:

$$\bigvee_{i\in I}(X_i, Y_i) = \Big(\bigcup_{i\in I} X_i, \bigcup_{i\in I} Y_i\Big) \quad \text{and} \quad \bigwedge_{i\in I}(X_i, Y_i) = \Big(\bigcap_{i\in I} X_i, \bigcap_{i\in I} Y_i\Big)$$

- $\mathcal{RS}$ is a complete sublattice of $\wp(U) \times \wp(U)$
- This is not easy to prove – it needs to show for a subset $\{({X_i}^{\blacktriangledown}, {X_i}^{\blacktriangle})\}_{i\in I} \subseteq RS$, that e.g. $(\bigcup_{i\in I} {X_i}^{\blacktriangledown}, \bigcup_{i\in I} {X_i}^{\blacktriangle})$ is a rough set, that is, there exists a set Z such that

$$Z^{\blacktriangledown} = \bigcup_{i\in I} {X_i}^{\blacktriangledown} \quad \text{and} \quad Z^{\blacktriangle} = \bigcup_{i\in I} {X_i}^{\blacktriangle}$$

The structure of $\mathcal{RS}$ in case of an equivalence relation

- $\wp(U) \times \wp(U)$ is (completely) distributive $\Rightarrow \mathcal{RS}$ is (completely) distributive
- The set of completely join-irreducible elements of $\mathcal{RS}$ is

$$\{(\emptyset, E(x)) : |E(x)| \geq 2\} \cup \{(E(x), E(x)) : x \in U\}$$

- The set of atoms of $\mathcal{RS}$ is

$$\{(\emptyset, E(x)) : |E(x)| \geq 2\} \cup \{(\{x\}, \{x\}) : E(x) = \{x\}\}$$

- $\mathcal{RS}$ is *spatial*, but not *atomistic*
- $\mathcal{RS}$ is not complemented, so it is not a Boolean lattice nor an ortholattice

Regular double Stone algebras

- In a bounded lattice L, x^* is a **pseudocomplement** of x, if $x \wedge x^* = 0$ and $x \wedge a = 0$ implies $a \leq x^*$ (unique)
- $(X^{\blacktriangledown}, X^{\blacktriangle})^* = (X^{\blacktriangle c}, X^{\blacktriangle c})$
- **Dual pseudocomplement** x^+: $x \vee x^+ = 1$ and $x \vee a = 1$ implies $a \geq x^+$
- $(X^{\blacktriangledown}, X^{\blacktriangle})^+ = (X^{\blacktriangledown c}, X^{\blacktriangledown c})$
- **Double Stone algebra**: $x^* \vee x^{**} = 1$ and $x^+ \wedge x^{++} = 0$
- A double Stone algebra is **regular** if $x^* = y^*$ and $x^+ = y^+$ imply $x = y$.
- If determined by an equivalence, $\mathcal{RS}$ forms a regular double Stone algebra
- Regular double Stone algebras can be identified with *3-valued Łukasiewicz–Moisil algebras* and *semi-simple Nelson algebras*

Rough sets determined by equivalences

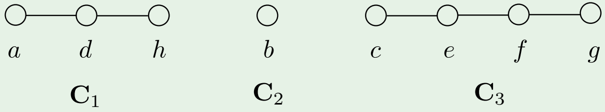
- $\mathcal{RS} \cong \mathbf{2}^I \times \mathbf{3}^J$, where I = set of singleton E-classes and J = set of non-singleton E-classes.

Remark

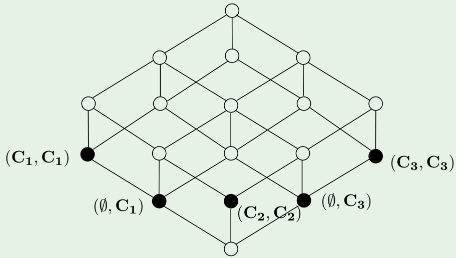
- The identity relation Id_U of U can be seen to represent *complete knowledge* in the sense that each element has a full identity, that is, every element can be discerned from the others.
- $(X^{\blacktriangledown}, X^{\blacktriangle}) = (X, X)$ for all $X \subseteq U$.
- This means that $\mathcal{RS}$ can be identified with $\wp(U)$, and $\mathcal{RS} \cong \wp(U) \cong \mathbf{2}^U$ in case $E = \mathrm{Id}_U$.

Example

Let R be the following equivalence on $U = \{a, b, c, d, e, f, g, h\}$



The rough set algebra $\mathcal{RS}$ is:



$$\cong \mathbf{2} \times \mathbf{3} \times \mathbf{3}$$

Some essential articles for equivalences

[1] Jacek Pomykała and Janusz A. Pomykała, The Stone algebra of rough sets, Bulletin of Polish Academy of Sciences. Mathematics 36 (1988), 495–512.

[2] Mai Gehrke and Elbert Walker, On the structure of rough sets, Bulletin of Polish Academy of Sciences. Mathematics 40 (1992), 235–245.

[3] Stephen D. Comer, On connections between information systems, rough sets, and algebraic logic, Algebraic Methods in Logic and Computer Science, Banach Center Publications, no. 28, 1993, pp. 117–124.

Rough sets defined by quasiorders

We will consider results from these articles:

[1] Jouni Järvinen, Sándor Radeleczki, and Laura Veres, *Rough sets determined by quasiorders*, Order **26** (2009), 337–355

[2] Jouni Järvinen and Sándor Radeleczki, *Representation of Nelson algebras by Rough Sets Determined by Quasiorders*, Algebra Universalis **66** (2011), 163–179.

[3] Jouni Järvinen, Piero Pagliani, Sándor Radeleczki, *Information completeness in Nelson algebras of rough sets induced by quasiorders*, Studia Logica **101** (2013), 1073–1092.

[4] Jouni Järvinen and Sándor Radeleczki, Monteiro spaces and rough sets determined by quasiorder relations: Models for Nelson algebras, Fundamenta Informaticae **131** (2014) 205–215.

$\mathcal{RS}$ induced by a quasiorder

Theorem

If R is a quasiorder on a non-empty set U, then $\mathcal{RS}$ is a complete sublattice of $\wp(U) \times \wp(U)$, that is,

$$\bigvee_{i \in I} (X_i^{\blacktriangledown}, X_i^{\blacktriangle}) = \Big(\bigcup_{i \in I} X_i^{\blacktriangledown}, \bigcup_{i \in I} X_i^{\blacktriangle} \Big)$$

and

$$\bigwedge_{i \in I} (X_i^{\blacktriangledown}, X_i^{\blacktriangle}) = \Big(\bigcap_{i \in I} X_i^{\blacktriangledown}, \bigcap_{i \in I} X_i^{\blacktriangle} \Big)$$

Since $\mathcal{RS}$ is a complete sublattice of $\wp(U) \times \wp(U)$, we may write:

Corollary

$\mathcal{RS}$ is a completely distributive lattice

In case of quasiorders, $\mathcal{RS}$ is algebraic

An element x of a complete lattice L is **compact** if for every $S \subseteq L$, $x \leq \bigvee S \implies x \leq \bigvee F$ for some finite $F \subseteq S$.

A complete lattice L is **algebraic**, if its every element is a join of compact elements.

Example

(a) $(\wp(U), \subseteq)$ is an algebraic lattice. Within this complete lattice, the compact elements are exactly the finite sets.
(b) $\wp(U) \times \wp(U)$ is an algebraic lattice, because the product of algebraic lattices is algebraic.

Because any complete sublattice of an algebraic lattice is algebraic, we may write:

Corollary

$\mathcal{RS}$ is an algebraic lattice.

Properties of algebraic lattices

For any lattice L, the following are known to be equivalent:

(a) L is isomorphic to an **Alexandrov topology**

(b) L is algebraic and completely distributive

(c) L is distributive and doubly algebraic, that is, also the dual L^d of L is algebraic

(d) L is algebraic, distributive and spatial

- Since $\mathcal{RS}$ is algebraic and completely distributive, it has the properties (a)–(d)
- In particular, $\mathcal{RS}$ is isomorphic to some Alexandrov topology
- How to get this Alexandrov topology we will find later
- $\mathcal{RS}$ forms a Heyting algebra.

Completely join-irreducible elements

Proposition

Let $\mathcal{RS}$ be determined by a quasiorder.

(a) *The set of completely join-irreducible elements of $\mathcal{RS}$ is*

$$\mathcal{J} = \{(\emptyset, \{x\}^{\blacktriangle}) \mid |R(x)| \geq 2\} \cup \{(R(x), R(x)^{\blacktriangle}) \mid x \in U\}.$$

(b) *The lattice $\mathcal{RS}$ is spatial*

Kleene algebras

A **Kleene algebra** is a structure $(A, \vee, \wedge, \sim, 0, 1)$ such that A is a 0,1-bounded distributive lattice and for all $x, y \in A$:

(K1) $\sim\sim x = x$

(K2) $x \leq y$ if and only if $\sim y \leq \sim x$

(K3) $x \wedge \sim x \leq y \vee \sim y$

A bounded distributive lattice A with $\sim$ satisfying (K1) and (K2) is a **De Morgan algebra**

Proposition

The algebra $\mathbb{RS} = (RS, \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))$ *is a Kleene algebra, where* $\sim$ *is defined*

$$\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{c\blacktriangledown}, X^{c\blacktriangle}) = (X^{\blacktriangle c}, X^{\blacktriangledown c})$$

Constructive logic with strong negation (Nelson logic) I

- **Constructive logic with strong negation** was introduced by Nelson (1949) and independently by Markov (1950). It is often called simply as **Nelson logic**.
- It is an extension of the intuitionistic propositional logic by **strong negation** $\sim$.
- There are generally two different ways to *refute* a sentence A.
- One way is by *reductio ad absurdum*: by proving that A implies absurdum. This role of negation is played both by the intuitionistic negation and by the classical negation
- $\neg A$ is defined to be $A \to \bot$

Constructive logic with strong negation (Nelson logic) II

- Another way to refute A is to construct a counterexample of A. The intuitive reading of $\sim A$ is "a counterexample of A".
- Sentence A may have many counterexamples and each of them have to contradict A. For instance, a counterexample of the sentence "This apple is red" is for instance "This apple is green" or "This apple is yellow"
- Axioms can be interpreted as "algorithms" of constructing counterexamples of compound sentences by means of given counterexamples of their components.
- The name *strong negation* comes from the fact that the formula $\sim A \to \neg A$ is a theorem of the logic.

Nelson logic – or Constructive logic with strong negation

(N1) $\sim A \to (A \to B)$
a counterexample of A contradicts A, that is, $A \wedge \sim A$ implies everything

(N2) $\sim(A \to B) \leftrightarrow A \wedge \sim B$
a counterexample of $A \to B$ can be constructed by the conjunction of A with a counterexample of B

(N3) $\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B$
a counterexample of a conjunction is a disjunction of counterexamples of its components

(N4) $\sim(A \vee B) \leftrightarrow \sim A \wedge \sim B$
a counterexample of a disjunction is a conjunction of counterexamples of its components

(N5) $\sim \neg A \leftrightarrow A$
A is a counterexample of $\neg A$

(N6) $\sim \sim A \leftrightarrow A$
A is a counterexample of a counterexample of A

Quasi-Nelson and Nelson algebras

- A **quasi-Nelson** algebra is a Kleene algebra $(A, \vee, \wedge, \sim, 0, 1)$ such that for each pair a and b of its elements, the element

$$a \Rightarrow \sim a \vee b$$

exists. Here $\Rightarrow$ denotes the Heyting implication in $(A, \leq)$:

$$c \leq a \Rightarrow b \text{ iff } a \wedge c \leq b$$

- This element is denoted $a \rightarrow b$ and called **weak relative pseudocomplement**. Hence,

$$c \leq a \rightarrow b \text{ iff } a \wedge c \leq \sim a \vee b$$

- Therefore, every Kleene algebra whose underlying lattice is a Heyting algebra forms a quasi-Nelson algebra.

Proposition

$\mathbb{RS} = (RS, \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))$ *is a quasi-Nelson algebra.*

Nelson algebras of rough sets determined by quasiorders

A **Nelson algebra** is a quasi-Nelson algebra $(A, \vee, \wedge, \sim, \to, 0, 1)$ satisfying:

$$(a \wedge b) \to c = a \to (b \to c)$$

Theorem

For any quasiorder, $(RS, \vee, \wedge, \to, \sim, 0, 1)$ *is a Nelson algebra such that:*

$$
\begin{aligned}
(X^{\blacktriangledown}, X^{\blacktriangle}) \vee (Y^{\blacktriangledown}, Y^{\blacktriangle}) &= (X^{\blacktriangledown} \cup Y^{\blacktriangledown}, X^{\blacktriangle} \cup Y^{\blacktriangle}) \\
(X^{\blacktriangledown}, X^{\blacktriangle}) \wedge (Y^{\blacktriangledown}, Y^{\blacktriangle}) &= (X^{\blacktriangledown} \cap Y^{\blacktriangledown}, X^{\blacktriangle} \cap Y^{\blacktriangle}) \\
\sim(X^{\blacktriangledown}, X^{\blacktriangle}) &= (X^{c\blacktriangledown}, X^{c\blacktriangle}) = (X^{\blacktriangle c}, X^{\blacktriangledown c}) \\
(X^{\blacktriangledown}, X^{\blacktriangle}) \to (Y^{\blacktriangledown}, Y^{\blacktriangle}) &= ((X^{\blacktriangledown c} \cup Y^{\blacktriangledown})^{\blacktriangledown}, X^{\blacktriangledown c} \cup Y^{\blacktriangle}) \\
0 &= (\emptyset, \emptyset) \\
1 &= (U, U)
\end{aligned}
$$

Weak negation and semi-simple Nelson algebras

- In each Nelson algebra, an operation $\neg$ can be defined as $\neg a = a \to 0$. The negation $\neg$ is called **weak negation**.
- A Nelson algebra is **semi-simple** if $a \vee \neg a = 1$
- It is known that $\mathcal{RS}$ defines a semi-simple Nelson algebra $\iff$ $\mathcal{RS}$ is defined by an equivalence

Kleene algebra defined on an algebraic lattice

If a Kleene algebra $\mathbb{A} = (A, \vee, \wedge, \sim, 0, 1)$ is defined on an algebraic lattice, then actually the underlying lattice A is doubly algebraic and distributive. Thus, A is isomorphic to an Alexandrov topology.

Then, we may define for any $j \in \mathcal{J}$ the element

$$g(j) = \bigwedge \{x \in A \mid x \not\leq \sim j\ \}$$

The map $g \colon \mathcal{J} \to \mathcal{J}$ satisfies:

(J1) if $x \leq y$, then $g(y) \leq g(x)$

(J2) $g(g(x)) = g(x)$

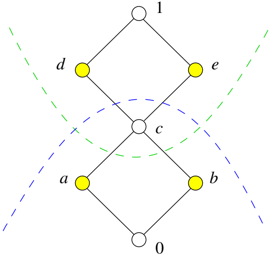
(J3) $x \leq g(x)$ or $g(x) \leq x$

(J4) $x, y \leq g(x), g(y)$ implies that there is $z \in \mathcal{J}$ such that

$$x, y \leq z \leq g(x), g(y)$$

Example

- Let us consider the Kleene algebra such that:
  $\sim 0 = 1$, $\sim a = e$, $\sim b = d$, $\sim c = c$
- Because the algebra is finite and distributive, it defines a Heyting algebra and so it forms a quasi-Nelson algebra.
- $\mathcal{J} = \{a, b, d, e\}$ and the map g is such that $g(a) = d$ and $g(b) = e$

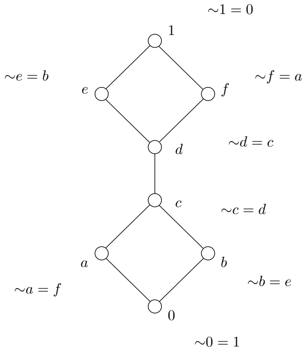


- Now $a, b \leq g(a), g(b)$, but there exists no $k \in \mathcal{J}$ such that $a, b \leq k \leq g(a), g(b) \implies$ This is not a Nelson algebra

Represention theorem

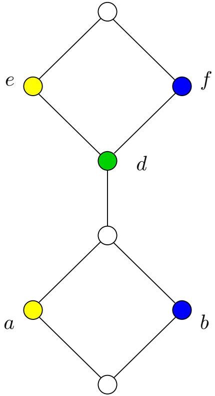
Theorem

If $\mathbb{A}$ is a Nelson algebra defined on an algebraic lattice, then there exists a set U and a quasiorder R on U such that $\mathbb{A} \cong \mathbb{RS}$.



For instance, $a \to b := a \Rightarrow (\sim a \vee b) = a \Rightarrow (f \vee b) = a \Rightarrow f = 1$, where $x \Rightarrow y := \bigvee \{z \mid z \wedge x \leq y\}$ is the Heyting implication.

Example of the construction



$g(f) = b$

$g(e) = a$

$g(d) = d$

$g(b) = f$

$g(a) = e$

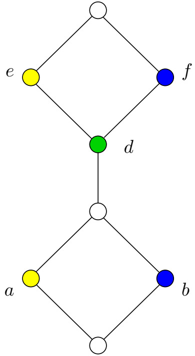
Example of the construction

We define a mapping $\rho\colon \mathcal{J} \to \mathcal{J}$:

$$\rho(j) = \begin{cases} j & \text{if } j \leq g(j) \\ g(j) & \text{otherwise} \end{cases}$$

In terms of ρ, we define a quasiorder R on $U = \mathcal{J}$ by

$$x \, R \, y \iff \rho(x) \leq \rho(y).$$



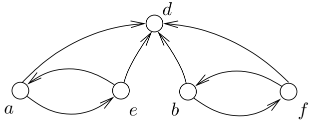
$\rho(f) = b$

$\rho(e) = a$

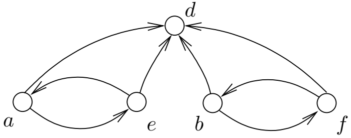
$\rho(d) = d$

$\rho(b) = b$

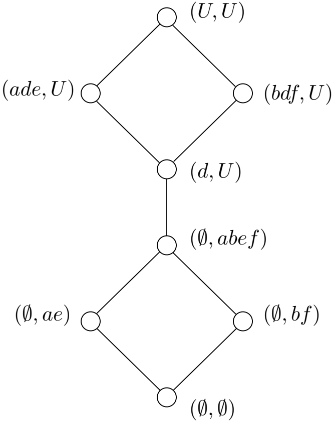
$\rho(a) = a$



Example of the construction



The relation R



$\mathcal{RS}$

Monteiro spaces and Alexandrov topologies of rough sets defined by quasiorders

Let $\mathcal{M} = (X, \leq, g)$ be a structure such that $(X, \leq)$ is a partially ordered set and g is a map on X satisfying:

(J1) if $x \leq y$, then $g(y) \leq g(x)$,

(J2) $g(g(x)) = g(x)$,

(J3) $x \leq g(x)$ or $g(x) \leq x$,

(J4) if $x, y \leq g(x), g(y)$, then there is $z \in X$ such that $x, y \leq z \leq g(x), g(y)$.

$\mathcal{M}$ is called a **Monteiro space**.

Proposition

Let $\mathbb{A}$ be a Nelson algebra defined on an algebraic lattice. If we define an order $\lhd$ on $\mathcal{J}$ by setting

$$x \lhd y \iff y \leq x \text{ in } A,$$

then $(\mathcal{J}, \lhd, g)$ is a Monteiro space.

Results by Vakarelov (1977)[1]

- For an ordered set $(X, \leq)$, we denote by $\mathcal{U}(X)$ the set of all **upward-closed subsets** of X.
- $\mathcal{U}(X)$ is an Alexandrov topology. It forms also a $\mathbf{T_0}$**-space**: for $x \neq y$, there is an open set which contains one of these points, but not the other.
- Each Monteiro space $\mathcal{M} = (X, \leq, g)$ defines a Nelson algebra

  $$(\mathcal{U}(X), \cup, \cap, \to, \sim, \emptyset, X),$$

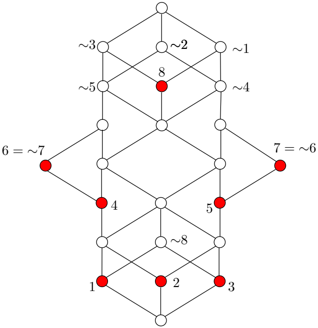
  where:

  $$\sim A = \{x \in X \mid g(x) \notin A\} \quad \text{and} \quad A \to B = A \Rightarrow (\sim A \cup B)$$

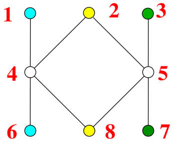
- Above $\Rightarrow$ is the Heyting implication of $\mathcal{U}(X)$

[1] Dimiter Vakarelov, *Notes on N-lattices and constructive logic with strong negation*, Studia Logica **36** (1977), 109–125.

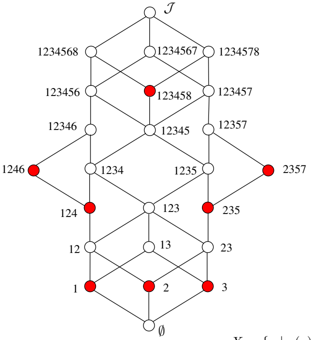
Example



Nelson algebra $\mathcal{A}$



$(\mathcal{J}, \triangleleft)$



$\sim X = \{a \mid g(a) \notin X\}$

Nelson algebra $\mathcal{U}(J)$

Proposition

The following structures can be considered equivalent, because they determine each other "up-to-isomorphism":

(i) *Rough sets by quasiorders*

(ii) *Nelson algebras defined on algebraic lattices*

(iii) *Nelson algebras defined on* T_0*-spaces that are Alexandrov topologies*

(iv) *Monteiro spaces*

Last results for quasiorder-based rough sets

Proposition

Let $\mathbb{A}$ be **any** *Nelson algebra. Then, there exists a set U and a quasiorder R on U such that $\mathbb{A}$ is isomorphic to a subalgebra of $\mathbb{RS}$.*

Theorem

Let ϕ be a formula of Nelson logic. TFAE:

1. *ϕ is a theorem*
2. *ϕ is valid in every finite rough set-based Nelson algebra determined by a quasiorder.*

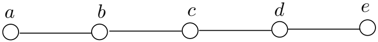
Rough sets determined by tolerances

The considered results are from the article:

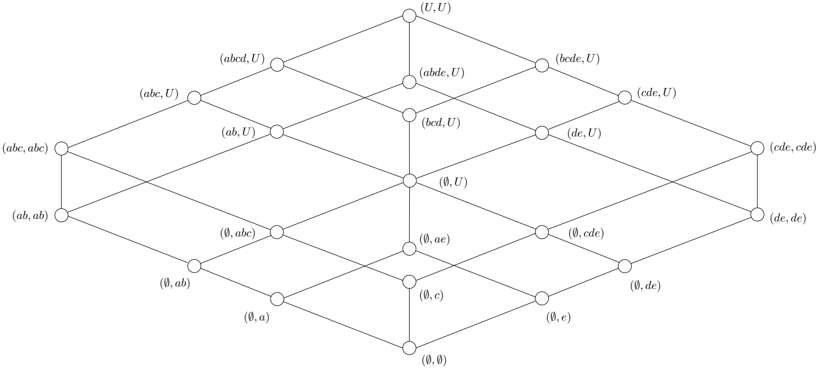
[1] Jouni Järvinen and Sándor Radeleczki, *Rough sets determined by tolerances*, Approximate Reasoning **55** (2014), 1419–1438

Rough sets determined by tolerances

Let us consider the following tolerance:



$\mathcal{RS}$ is not necessarily a lattice:



Rough sets determined by tolerances

A **complete subdirect product** $\mathcal{L}$ of an indexed family of complete lattices $\{L_i\}_{i \in I}$ is a complete sublattice of the direct product $\prod_{i \in I} L_i$ such that the canonical projections π_i are all surjective, that is, $\pi_i(\mathcal{L}) = L_i$.

The projections π_i are complete lattice homomorphisms, that is, they preserve all meets and joins.

Proposition

$\mathcal{RS}$ is a complete lattice if and only if it is a complete subdirect product of the complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.

Lattice operations in $\mathcal{RS}$ I

Let R be a tolerance on U. Recall that

- $\wp(U)^{\blacktriangledown}$ is a complete lattice such that for $\mathcal{H} \subseteq \wp(U)$:

$$\bigvee_{X \in \mathcal{H}} X^{\blacktriangledown} = \Big(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown} \Big)^{\blacktriangle\blacktriangledown} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}$$

- $\wp(U)^{\blacktriangle}$ is a complete lattice such that for $\mathcal{H} \subseteq \wp(U)$:

$$\bigvee_{X \in \mathcal{H}} X^{\blacktriangle} = \bigcup_{X \in \mathcal{H}} X^{\blacktriangle} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle} \Big)^{\blacktriangledown\blacktriangle}$$

Lattice operations in $\mathcal{RS}$ II

We have that if $\mathcal{RS}$ is a complete lattice, then it **must** be a complete sublattice of the product $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$

Let $({X_i}^{\blacktriangledown}, {X_i}^{\blacktriangle}) \subseteq \mathcal{RS}$. The meet and join are defined by:

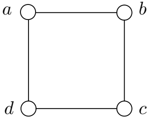
$$\bigwedge_{i \in I} ({X_i}^{\blacktriangledown}, {X_i}^{\blacktriangle}) = \Big(\bigcap_{i \in I} {X_i}^{\blacktriangledown}, (\bigcap_{i \in I} {X_i}^{\blacktriangle})^{\blacktriangledown\blacktriangle} \Big)$$

and

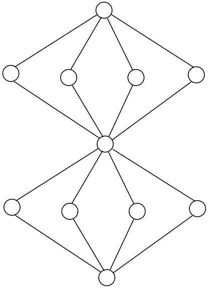
$$\bigvee_{i \in I} ({X_i}^{\blacktriangledown}, {X_i}^{\blacktriangle}) = \Big((\bigcup_{i \in I} {X_i}^{\blacktriangledown})^{\blacktriangle\blacktriangledown}, \bigcup_{i \in I} {X_i}^{\blacktriangle} \Big)$$

Example

Let $U = \{a, b, c, d\}$ and let R be the following tolerance



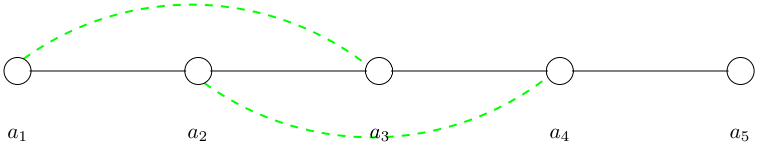
Then RS has the following 11 elements: $(\emptyset, \emptyset)$, $(\emptyset, \{a, b, c\})$, $(\emptyset, \{a, b, d\})$, $(\emptyset, \{a, c, d\})$, $(\emptyset, \{b, c, d\})$, $(\emptyset, U)$, $(\{a\}, U)$, $(\{b\}, U)$, $(\{c\}, U)$, $(\{d\}, U)$, (U, U)



The lattice is not distributive

A condition under which $\mathcal{RS}$ is a complete lattice

(C) For any R-path $(a_1, \ldots, a_5)$ of 5 elements, there exist $1 \leq i, j \leq 5$ such that $|i - j| \geq 2$ and $a_i \, R \, a_j$.



Theorem

If R is a tolerance satisfying (C), then $\mathcal{RS}$ is a complete lattice.

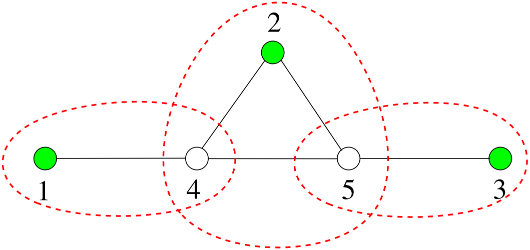
Rough sets determined by tolerances

Theorem

Let R be a tolerance on U. Then RS is an algebraic completely distributive lattice if and only if R is induced by an irredundant covering of U.

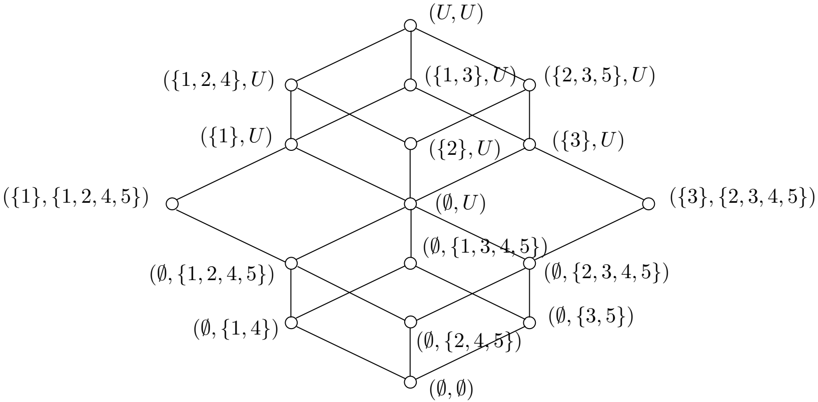
Example: tolerance induced by an irredundant covering

Let $U = \{1, 2, 3, 4, 5\}$ and suppose that R is the following tolerance on U:



The tolerance R is induced by the irredundant covering $\{R(1), R(2), R(3)\}$.

Example: tolerance induced by an irredundant covering



Rough sets determined by tolerances

Proposition

Let R be a tolerance induced by an irredundant covering of U. Then,

$$(\mathcal{RS}, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a Kleene algebra, where

$$\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{c\blacktriangledown}, X^{c\blacktriangle}) = (X^{\blacktriangle c}, X^{\blacktriangledown c}).$$

This algebra is always also a quasi-Nelson algebra – but a Nelson algebra only if the R is an equivalence.