Theoretical Foundations I: Structure of Rough Approximations

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Relations

- Let $U$ be a nonempty set, called often **universe** (or **universe of discourse**). This is the set of elements (or objects) we are interested in.

- Let $R$ be a **binary relation** on $U$ representing some knowledge about the elements in $U$. Binary relation $R$ consists of ordered pairs $(a, b)$ such that $a R b$.

- The relation $R$ is interpreted to represent some knowledge about the objects in $U$.

**Example**

Let us consider a datatable (database table, Excel table, etc.) representing some information about human beings. The table may contain columns such that **weight**, **height**, **age**, **gender**, **home town**, etc. Two objects are $R$-related if values for all the above attributes are the same.
Example

In the previous example, we defined the relation $R$ in such a way that two objects are $R$-related, if they have exactly the same values for all attributes.

We may also define a relation such that two objects are $R$-related if their values are “close enough”. For instance,

$$a R b \iff |\text{height}(a) - \text{height}(b)| \leq \varepsilon,$$

where $\varepsilon$ is a suitable threshold. In the next figure, $\varepsilon$ equals 3 cm.
Example

The statement “$a$ is preferred to $b$” is generally understood to mean that someone chooses $a$ over $b$. In this case, the universe $U$ consists of different choices and $a R b$ tells that $a$ is preferred over $b$. For instance, someone could say that: “I prefer sleeping over running”.
Rough approximations based on binary relations

Let $R$ be a binary relation on $U$, and let us denote for all $x \in U$, 

$$R(x) = \{ y \in U \mid x R y \}$$

The **upper approximation** of $X \subseteq U$ is 

$$X^\Delta = \{ x \in U \mid R(x) \cap X \neq \emptyset \}$$

and the **lower approximation** of $X$ is 

$$X^\nabla = \{ x \in U \mid R(x) \subseteq X \}$$

The set $B(X) = X^\Delta \setminus X^\nabla$ is the **boundary** of $X$.
Rough approximations

The lower approximation of $X^\downarrow$ can be viewed as the set of elements that certainly are in $X$ when observed through the knowledge $R$, because all elements $R$-related to them are in $X$.

The upper approximation $X^\uparrow$ can be viewed as the set of elements that possible are in $X$, because in $X$ is at least one element $R$-related to them.
Proposition

If $R$ is a binary relation on $U$, then following assertions hold.

(a) The maps $\nabla$ and $\Delta$ are mutually dual, i.e.
    $X^{\nabla c} = X^{c\Delta}$ and $X^{\Delta c} = X^{c\nabla}$

(b) The boundary of any set is equal to the boundary of its complement.

(c) The maps $\nabla$ and $\Delta$ are order-preserving.

Proof.

(a) $x \in X^{\nabla c} \iff x \notin X^{\nabla} \iff R(x) \nsubseteq X \iff R(x) \cap X^c \neq \emptyset \iff x \in X^{c\Delta}$. Further, $X^{\Delta c} = X^{cc\Delta c} = X^{c\nabla cc} = X^{c\nabla}$.

(b) $B(X) = X^{\Delta} \setminus X^{\nabla} = X^{\Delta} \cap X^{\nabla c} = X^{c\nabla c} \cap X^{c\Delta} = X^{c\Delta} \setminus X^{c\nabla} = B(X^c)$.

(c) Suppose $X \subseteq Y$. If $x \in X^{\nabla}$, then $R(x) \subseteq X \subseteq Y$, i.e. $x \in Y^{\nabla}$. If $x \in X^{\Delta}$, then $R(x) \cap Y \supseteq R(x) \cap X \neq \emptyset$, i.e. $x \in Y^{\Delta}$.
We denote
\[
\wp(U)^\triangledown = \{X^\triangledown \mid X \subseteq U\} \quad \text{and} \quad \wp(U)^\triangle = \{X^\triangle \mid X \subseteq U\}.
\]

**Proposition**

The ordered sets \((\wp(U)^\triangle, \subseteq)\) and \((\wp(U)^\triangledown, \subseteq)\) are dually isomorphic.

**Proof.**

We show that the map \(\phi: X^\triangle \mapsto X^{c\triangledown}\) is an order-isomorphism between \((\wp(U)^\triangle, \subseteq)\) and \((\wp(U)^\triangledown, \supseteq)\).

\[
X^\triangle \subseteq Y^\triangle \iff \phi(X^\triangle) = X^{c\triangledown} = X^{\triangle c} \supseteq Y^{\triangle c} = Y^{c\triangledown} = \phi(Y^\triangle).
\]

Thus, \(\phi\) is an order-embedding.

If \(X^\triangledown \in \wp(U)^\triangledown\), then \(\phi(X^{c\triangle}) = X^{cc\triangledown} = X^{\triangledown}\), i.e., \(\phi\) is onto. \(\square\)
Example

Let $U = \{a, b, c, d\}$.

$(\wp(U)\uparrow, \subseteq)$ and $(\wp(U)\downarrow, \subseteq)$ seem to be lattices (we will study in detail what kind of lattices these are).

$(\wp(U)\downarrow)$ and $(\wp(U)\uparrow)$ are not distributive, because they contain $M_3$ as a sublattice.

These lattices are not complemented.
We denote by $R^{-1}$ the inverse relation of $R$ and

$$R^{-1}(x) = \{ y \mid y R x \}.$$

We define

$$X^{\triangle} = \{ x \in U \mid R^{-1}(x) \cap X \neq \emptyset \}$$

and

$$X^{\triangledown} = \{ x \in U \mid R^{-1}(x) \subseteq X \}.$$

Note that

$$\{ x \}^{\Delta} = \{ y \mid R(y) \cap \{ x \} \neq \emptyset \} = \{ y \mid x \in R(y) \} = R^{-1}(x)$$

Similarly,

$$\{ x \}^{\triangle} = R(x)$$
Galois connections are pairs of maps which enable us to move back and forth between two ordered sets.

Galois connections tie different structures firmly and when a Galois connection is found between two structures, we immediately know that they have much in common.

After an element is mapped to the other structure and back, a certain stability is reached in such a way that further mappings give the same results.

We will show that the pairs \((\uparrow, \downarrow)\) and \((\triangle, \nabla)\) form Galois connections. Several observations and properties of rough approximations follow from this.
Definition ("flip-flop" property)

For two partially ordered sets \((P, \leq)\) and \((Q, \leq)\), a pair \((f, g)\) of maps \(f: P \to Q\) and \(g: Q \to P\) is called a Galois connection between \(P\) and \(Q\) if for all \(p \in P\) and \(q \in Q\),

\[
f(p) \leq q \iff p \leq g(q).
\]

Such a mapping \(f\) is sometimes called residuated mapping. The mapping \(g\) is called the residual mapping of \(f\).

Lemma

The pair \((f, g)\) is a Galois connection between \(P\) and \(Q\) iff

(a) \(p \leq (g \circ f)(p)\) for all \(p \in P\) and \((f \circ g)(q) \leq q\) for all \(q \in Q\)

(b) the maps \(f\) and \(g\) are order-preserving
Galois connections I

Let \((f, g)\) be a Galois connection between two complete lattices \(P\) and \(Q\).

1. \(f \circ g \circ f = f\) and \(g \circ f \circ g = g\).

2. The map \(g \circ f\) is a (lattice-theoretical) closure operator on \(P\) (extensive, order-preserving, idempotent) and the set of \(g \circ f\)-closed elements is \(g(Q)\), that is, \((g \circ f)(P) = g(Q)\).

3. The map \(f \circ g\) is a (lattice-theoretical) interior operator on \(Q\) (inflationary, order-preserving, idempotent) and the set of \(f \circ g\)-open elements is \(f(P)\), that is, \((f \circ g)(Q) = f(P)\).

4. The map \(f\) is a complete join-morphism and \(g\) is a complete meet-morphism, that is,

\[
 f(\bigvee S) = \bigvee f(S) \quad \text{and} \quad g(\bigwedge T) = \bigwedge g(T)
\]

for \(S \subseteq P\) and \(T \subseteq Q\).
5. The image sets \( f(P) \) and \( g(Q) \) are order-isomorphc.

6. The ordered set \( f(P) \) is a complete lattice such that for all \( S \subseteq f(P) \) \( (\subseteq Q) \),

\[
\bigvee S = \bigvee_Q S \quad \text{and} \quad \bigwedge S = f\left(g\left(\bigwedge_Q S\right)\right) = f\left(\bigwedge_P g(S)\right).
\]

7. The ordered set \( g(Q) \) is a complete lattice such that for all \( S \subseteq g(Q) \) \( (\subseteq P) \),

\[
\bigvee S = g\left(f\left(\bigvee_P S\right)\right) = g\left(\bigvee_Q f(S)\right) \quad \text{and} \quad \bigwedge S = \bigwedge_P S.
\]
Galois connections of rough approximations

The ordered set \((\wp(U), \subseteq)\) is a complete lattice such that

\[
\bigvee \mathcal{H} = \bigcup \mathcal{H} \quad \text{and} \quad \bigwedge \mathcal{H} = \bigcap \mathcal{H}
\]

for all \(\mathcal{H} \subseteq \wp(U)\).

**Proposition**

*For any binary relation \(R\) on \(U\), the pairs \((\blacktriangledown, \blacktriangledown)\) and \((\blacktriangle, \blacktriangle)\) are order-preserving Galois connections on \((\wp(U), \subseteq)\).*

**Proof.**

As noted, the maps \(X \mapsto X^\blacktriangledown\) and \(X \mapsto X^\blacktriangle\) are order-preserving. If \(x \in X^{\blacktriangledown}\), there exists \(y \in X^{\blacktriangle}\) such that \((x, y) \in R\). Because \(y \in X^{\blacktriangle}\) and \((y, x) \in R^{-1}\), we have \(x \in X\). Hence, \(X^{\blacktriangledown} \subseteq X\). This also gives \(X^{\blacktriangle \blacktriangledown} = X^{c^{\blacktriangledown}} \subseteq X^{c}\), i.e., \(X \subseteq X^{\blacktriangle \blacktriangledown}\). \(\blacksquare\)
What this then means? I

1. \( X^\land \lor = X^\land, X^\lor \land = X^\lor, X^\lor \land = X^\lor, X^\land \lor = X^\lor \)

2 a. The map \( X \mapsto X^\land \lor \) is a closure operator. The set of closed sets is \( \wp(U)^\land \), i.e. \( \{ X^\land \lor \mid X \subseteq U \} = \wp(U)^\land \).

2 b. The map \( X \mapsto X^\lor \land \) is a closure operator. The set of closed sets is \( \wp(U)^\lor \), i.e. \( \{ X^\lor \land \mid X \subseteq U \} = \wp(U)^\lor \).

3 a. The map \( X \mapsto X^{\land} \lor \) is an interior operator. The set of open sets is \( \wp(U)^\land \), i.e. \( \{ X^{\land} \lor \mid X \subseteq U \} = \wp(U)^\land \).

3 b. The map \( X \mapsto X^{\lor} \land \) is an interior operator. The set of open sets is \( \wp(U)^\lor \), i.e. \( \{ X^{\lor} \land \mid X \subseteq U \} = \wp(U)^\lor \).
What this then means? II

4 a. For $\mathcal{H} \subseteq \wp(U)$:

$\left( \bigcup_{X \in \mathcal{H}} X \right)^\triangle = \bigcup_{X \in \mathcal{H}} X^\triangle$ and $\left( \bigcup_{X \in \mathcal{H}} X \right)^\lozenge = \bigcup_{X \in \mathcal{H}} X^\lozenge$

Note that this implies that $X^\triangle = \bigcup_{x \in X} \{x\}^\triangle = \bigcup_{x \in X} R^{-1}(x)$
and $X^\lozenge = \bigcup_{x \in X} \{x\}^\lozenge = \bigcup_{x \in X} R(x)$

4 b. $\left( \bigcap_{X \in \mathcal{H}} X \right)^\lozenge = \bigcap_{X \in \mathcal{H}} X^\lozenge$ and $\left( \bigcap_{X \in \mathcal{H}} X \right)^\triangle = \bigcap_{X \in \mathcal{H}} X^\triangle$

5. $\wp(U)^\triangle \cong \wp(U)^\lozenge$ and $\wp(U)^\lozenge \cong \wp(U)^\triangle$
6 a. The ordered set \((\wp(U)^\sqcup, \subseteq)\) is a complete lattice such that

\[
\bigvee_{X \in \mathcal{H}} X^\sqcup = \bigcup_{X \in \mathcal{H}} X^\sqcup \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^\sqcup = \left( \bigcap_{X \in \mathcal{H}} X^\sqcup \right)^\downarrow
\]

6 b. The ordered set \((\wp(U)^\triangle, \subseteq)\) is a complete lattice such that

\[
\bigvee_{X \in \mathcal{H}} X^\triangle = \bigcup_{X \in \mathcal{H}} X^\triangle \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^\triangle = \left( \bigcap_{X \in \mathcal{H}} X^\triangle \right)^\nabla
\]
7 a. The ordered set \((\wp(U)^\downarrow, \subseteq)\) is a complete lattice such that
\[
\bigwedge_{X \in H} X^\downarrow = \bigcap_{X \in H} X^\downarrow \quad \text{and} \quad \bigvee_{X \in H} X^\downarrow = \left( \bigcup_{X \in H} X^\downarrow \right)^\uparrow
\]

7 b. The ordered set \((\wp(U)^\uparrow, \subseteq)\) is a complete lattice such that
\[
\bigwedge_{X \in H} X^\uparrow = \bigcap_{X \in H} X^\uparrow \quad \text{and} \quad \bigvee_{X \in H} X^\uparrow = \left( \bigcup_{X \in H} X^\uparrow \right)^\downarrow
\]
Different types of relations

A binary relation $R$ on $U$ is said to be:
  - **left-total** if for all $x \in U$, the exists $y \in U$ such that $x R y$.
  - **reflexive** if for all $x \in U$, $x R x$.
  - **symmetric** if $x R y$ implies $y R x$.
  - **antisymmetric** if $x R y$ and $y R x$ imply $x = y$.
  - **transitive** if $x R y$ and $y R z$ imply $x R z$.
  - a **tolerance** if it is reflexive and symmetric
  - a **quasiorder (or a preorder)** if it is reflexive and transitive
  - a **partial order** if it is reflexive, antisymmetric and transitive
  - an **equivalence** if it is reflexive, symmetric and transitive
Example

Let $E$ be an equivalence on $U$ such that $\{a, b\}$ and $\{c, d\}$ are $E$-equivalence classes.

We know that $X^\uparrow \cup Y^\uparrow = (X \cup Y)^\uparrow$, but $X^\uparrow \cap Y^\uparrow \supseteq (X \cap Y)^\uparrow$, and the inclusion can be proper!

Let $X = \{a, c\}$ and $Y = \{b, d\}$. Then $X^\uparrow = U$ and $Y^\uparrow = U$, and $X^\uparrow \cap Y^\uparrow = U$, but $(X \cap Y)^\uparrow = \emptyset^\uparrow = \emptyset$.

Analogously, we have $X^\downarrow \cap Y^\downarrow = (X \cap Y)^\downarrow$, but $X^\downarrow \cup Y^\downarrow \subseteq (X \cup Y)^\downarrow$.

Also this inclusion can be proper, because $X^\downarrow = \emptyset$, $Y^\downarrow = \emptyset$, and $X^\downarrow \cup Y^\downarrow = \emptyset$. But: $(X \cup Y)^\downarrow = U^\downarrow = U$. 

Correspondences: left-total relations

Proposition

If $R$ is a binary relation on $U$, then the following are equivalent:

(a) $R$ is left-total;
(b) $X\downarrow \subseteq X\uparrow$ for all $X \subseteq U$.

Proof.

(a) $\Rightarrow$ (b): Let $x \in X\downarrow$. Then $R(x) \subseteq X$, which gives $R(x) \cap X = R(x) \neq \emptyset$, i.e., $x \in X\uparrow$.

(b) $\Rightarrow$ (a): Assume that $R$ is not left-total, i.e. $R(x) = \emptyset$ for some $x \in U$. This means that $x \in X\downarrow$ and $x \notin X\uparrow$ for this particular $x$ and for any set $X \subseteq U$, a contradiction!
Correspondences: reflexive relations

Proposition

TFAE:
(a) \( R \) is reflexive;
(b) \( X \subseteq X^\uparrow \) for all \( X \subseteq U \);
(c) \( X^\downarrow \subseteq X \) for all \( X \subseteq U \).

Proof.

(a) \( \Rightarrow \) (b): If \( x \in X \), then \( x \in R(x) \cap X \neq \emptyset \), i.e. \( x \in X^\uparrow \).
(b) \( \Rightarrow \) (c): \( X^c \subseteq X^{c\uparrow} = X^{\downarrow^c} \) gives \( X^\downarrow \subseteq X \).
(c) \( \Rightarrow \) (a): If \( R \) is not reflexive, there is \( x \in U \) such that \( (x, x) \notin R \). Let us consider the set \( X = U \setminus \{x\} \). Now \( (x, y) \in R \) implies \( y \in X \). Thus, \( x \in X^\downarrow \) and \( x \notin X \), a contradiction! \( \square \)
Proposition

TFAE:

(a) $R$ is symmetric;
(b) $(\uparrow, \downarrow)$ is a Galois connection on $(\wp(U), \subseteq)$.

Proof.

(a) $\Rightarrow$ (b): If $R$ is symmetric, then $X^\uparrow = X^\Delta$ and $X^\downarrow = X^\nabla$ for all $X \subseteq U$. Recall that $(\uparrow, \downarrow)$ is a Galois connection.

(b) $\Rightarrow$ (a): If $R$ is not symmetric, then for some $x, y \in U$, $(x, y) \in R$, but $(y, x) \not\in R$. Let $X = \{x\}$. For all $z \in U$, $(y, z) \in R$ implies $z \not\in X$. This gives $y \not\in X^\uparrow$. Hence, $x \in X$ and $x \not\in X^\uparrow \downarrow$, a contradiction!
Correspondences: transitive relations

Proposition

TFAE:

(a) $R$ is transitive;
(b) $X^{\uparrow\uparrow} \subseteq X^{\uparrow}$ for all $X \subseteq U$;
(c) $X^{\downarrow} \subseteq X^{\downarrow\downarrow}$ for all $X \subseteq U$.

Proof.

(a) $\Rightarrow$ (b): Let $x \in X^{\uparrow\uparrow}$. There is $y \in X^{\uparrow}$ such that $(x, y) \in R$. Since $y \in X^{\uparrow}$, there is $z \in X$ such that $(y, z) \in R$. So, also $(x, z) \in R$ and $x \in X^{\uparrow}$.

(b) $\Rightarrow$ (c): $X^{\downarrow\downarrow c} = X^{c\uparrow\uparrow c} \subseteq X^{c\uparrow} = X^{\downarrow c}$, which gives $X^{\downarrow} \subseteq X^{\downarrow\downarrow}$.

(c) $\Rightarrow$ (a): If $R$ is not transitive, there are $x, y, z \in U$ such that $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$. Let $X = U \setminus \{z\}$. Then for all $w \in U$, $(x, w)$ implies $w \in X$. Thus, $x \in X^{\downarrow}$. Obviously, $y \notin X^{\downarrow}$ and hence $x \notin X^{\downarrow\downarrow}$, a contradiction! □
Correspondences for $\triangle$ and $\nabla$

- Note that $R$ is reflexive if and only if $R^{-1}$ is reflexive,
- Similar conditions hold also for symmetry and transitivity.
- We can state similar correspondences between $R$ and the operators $X \mapsto X^{\triangle}$ and $X \mapsto X^{\nabla}$.
- However, with left-/right-total relations we have to make the following exception:

\[
(\forall X \subseteq U) \ X^{\nabla} \subseteq X^{\triangle} \iff R^{-1} \text{ is left-total} \\
\iff R \text{ is right-total}
\]
Properties of rough approximations: tolerances

Let $R$ be a tolerance on $U$ and $X, Y \subseteq U$.

(a) $X \uparrow \subseteq X \subseteq X \downarrow$

(b) $(\uparrow, \downarrow)$ is an order-preserving Galois connection on $(\wp(U), \subseteq)$:

\[X \uparrow \subseteq Y \iff X \subseteq Y \downarrow\]

(c) $X \uparrow \downarrow \uparrow = X \uparrow$ and $X \downarrow \uparrow \downarrow = X \downarrow$.

Proposition

Let $(F, G)$ be a Galois connection on $(\wp(U), \subseteq)$. There exists a tolerance $R$ on $U$ such that $F$ equals $\uparrow$ and $G$ equals $\downarrow$ if and only if the following conditions hold for all $x, y \in U$:

(i) $x \in F(\{x\})$;

(ii) $x \in F(\{y\})$ implies $y \in F(\{x\})$. 
Lattice structures of approximations: **tolerances**

Let $R$ be a tolerance.

(a) $(\wp(U)\uparrow, \subseteq)$ forms a complete lattice such that for $H \subseteq \wp(U)$:

$$\bigvee_{X \in H} X^\uparrow = (\bigcup_{X \in H} X^\uparrow)^\uparrow$$

and

$$\bigwedge_{X \in H} X^\uparrow = \bigcap_{X \in H} X^\uparrow$$

(b) $(\wp(U)^\downarrow, \subseteq)$ forms a complete lattice such that for $H \subseteq \wp(U)$:

$$\bigvee_{X \in H} X^\downarrow = \bigcup_{X \in H} X^\downarrow$$

and

$$\bigwedge_{X \in H} X^\downarrow = (\bigcap_{X \in H} X^\downarrow)^\downarrow$$

(c) The maps $X^\uparrow \mapsto X^\uparrow^\downarrow$ and $X^\downarrow \mapsto X^\downarrow^\uparrow$ are isomorphisms between $\wp(U)^\uparrow$ and $\wp(U)^\downarrow$ — these are now also **self-dual**
Distributivity and modularity

A lattice is **distributive** if for all \( x, y, z \):

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z). \]

A lattice is distributive iff none of its sublattices is isomorphic to \( M_3 \) or \( N_5 \).

A **modular lattice** is a lattice that satisfies the condition:

\[ x \leq b \text{ implies } x \lor (a \land b) = (x \lor a) \land b. \]

A lattice \( L \) is modular iff none of its sublattices is isomorphic to \( N_5 \).
Example

The lattices $\wp(U)^\downarrow$ and $\wp(U)^\uparrow$ are not always distributive.
Example

Tolerance $R$

$\varnothing \cup \{a\} \cup \{d\} \cup \{a, b\} \cup \{c, d\}$

$\wp(U) \blacktriangleleft$

$\Longrightarrow$ The lattices $\wp(U) \blacktriangleleft$ and $\wp(U) \blacktriangleright$ are not always modular
Approximation lattices as ortholattices

An ortholattice is a bounded lattice equipped with an orthocomplementation:

\[(O1)\] \(x \leq y \) implies \(y \perp \leq x \perp\)

\[(O2)\] \(x \perp \perp = x\)

\[(O3)\] \(x \lor x \perp = 1\) and \(x \land x \perp = 0\)

Lemma

Let \(R\) be a tolerance

(a) \(\wp(U)\uparrow\) is an ortholattice such that \(\perp: X\uparrow \mapsto X\uparrow c\uparrow\)

(b) \(\wp(U)\downarrow\) is an ortholattice such that \(\top: X\downarrow \mapsto X\downarrow c\downarrow\)

Proposition

A complete lattice \(L\) forms an ortholattice if and only if there exists a set \(U\) and a tolerance \(R\) on \(U\) such that \(L \cong \wp(U)\downarrow \cong \wp(U)\uparrow\).
Irredundant coverings

A collection $\mathcal{H}$ of nonempty subsets of $U$ is called a covering of $U$ if $\bigcup \mathcal{H} = U$.

A covering $\mathcal{H}$ is irredundant if $\mathcal{H} \setminus \{X\}$ is not a covering for any $X \in \mathcal{H}$.

Each covering $\mathcal{H}$ of $U$ defines a tolerance $\bigcup \{X^2 \mid X \in \mathcal{H}\}$, called the tolerance induced by $\mathcal{H}$.

**Proposition**

*Let $R$ be a tolerance induced by a covering $\mathcal{H} \subseteq \wp(U)$. Then, the following assertions are equivalent:*

(a) $\mathcal{H}$ is an irredundant covering;

(b) $\mathcal{H} \subseteq \{R(x) \mid x \in U\}$
Example

Any tolerance $R$ on $U$ determines an undirected graph $\mathcal{G} = (U, R)$.

The family $\mathcal{H} = \{\{a, b, d, e\}, \{b, c, d, f\}, \{d, e, f, g\}\}$ induces $R$. This covering $\mathcal{H}$ is irredundant, because $R(a) = \{a, b, d, e\}$, $R(c) = \{b, c, d, f\}$, $R(g) = \{d, e, f, g\}$. 
Definition

1. A complete lattice $L$ satisfies the **join-infinite distributive law (JID)** if for any $S \subseteq L$ and $x \in L$,

$$x \land (\bigvee S) = \bigvee \{x \land y \mid y \in S\}. \quad \text{(JID)}$$

2. The **meet-infinite distributive law (MID)** is defined:

$$x \lor (\bigwedge S) = \bigwedge \{x \lor y \mid y \in S\}. \quad \text{(MID)}$$

3. A complete lattice $L$ is **completely distributive** if arbitrary joins distribute over arbitrary meets.

Proposition

For a tolerance $R$ on $U$, the isomorphic complete lattices $\wp(U)^\downarrow$ and $\wp(U)^\uparrow$ are completely distributive if and only if $R$ is induced by an irredundant covering of $U$. 
Definition

- A bounded lattice is **complemented** if every element $a$ has a complement $a'$: $a \lor a' = 1$ and $a \land a' = 0$.
- A complement is unique if the lattice is distributive.
- **Boolean lattice**: distributive and complemented lattice
- **Boolean algebra**: $(B, \lor, \land, \lnot, 0, 1)$

Remark

A distributive ortholattice is a Boolean lattice. Each Boolean lattice is trivially an ortholattice.
Blocks of a tolerance

- A nonempty subset $X$ of $U$ is an $R$-preblock if $X^2 \subseteq R$.
- An $R$-block is a maximal $R$-preblock.
- The relation $R$ is completely determined by its blocks, i.e., $a R b$ if and only if there exists a block $B$ such that $a, b \in B$.

Lemma

If $R$ is a tolerance induced by an irredundant covering $\mathcal{H}$, then

$$\mathcal{H} = \{ R(x) | R(x) \text{ is a block} \}.$$ 

For all $x \in U$, $R(x)$ is an $R$-block if and only if $R(x)$ is an $R$-preblock, i.e. a clique of the graph $\mathcal{G} = (U, R)$. 
Let $L$ be a lattice with a least element $0$. The lattice $L$ is **atomistic**, if any element of $L$ is the join of atoms below it. It is well known that a complete Boolean lattice is atomistic if and only if it is completely distributive.

**Proposition**

Let $R$ be a tolerance induced by an irredundant covering of $U$.

(a) $\wp(U)\uparrow$ and $\wp(U)\downarrow$ are atomistic Boolean lattices

(b) $At(\wp(U)\uparrow) = \{R(x) \mid R(x) \text{ is a block}\}$

(c) $At(\wp(U)\downarrow) = \{R(x)\downarrow \mid R(x) \text{ is a block}\}$
A **topological space** \((U, \mathcal{T})\) consists of a set \(U\) and a family \(\mathcal{T} \subseteq \wp(U)\) such that

1. \(\emptyset \in \mathcal{T}\) and \(U \in \mathcal{T}\),
2. \(X \cap Y \in \mathcal{T}\) for any sets \(X, Y \in \mathcal{T}\), and
3. \(\bigcup \mathcal{H} \in \mathcal{T}\) for any subfamily \(\mathcal{H} \subseteq \mathcal{T}\).

The family \(\mathcal{T}\) is called a **topology** on \(U\) and the members of \(\mathcal{T}\) are **open sets**. The complement of an open set is called a **closed set**.

An operator \(C : \wp(U) \rightarrow \wp(U)\) is a **Kuratowski closure operator** if for any \(X, Y \subseteq U\),

1. \(X \subseteq C(X)\),
2. \(C(C(X)) = C(X)\),
3. \(C(X \cup Y) = C(X) \cup C(Y)\), and
4. \(C(\emptyset) = \emptyset\).
If \( \mathcal{T} \) is a topology on \( U \), then the operator defined by

\[
C(X) = \bigcap \{B \mid X \subseteq B \text{ and } B \text{ is closed}\}
\]

is a Kuratowski closure operator.

Conversely, for a Kuratowski closure operator \( C \) on \( U \), the family

\[
\{C(X) \mid X \subseteq U\}
\]

determines a topological space whose closed sets are exactly these sets.

Kuratowski closure operators are in 1-to-1 correspondence with topologies.
Heyting algebras of topologies

- A **Heyting algebra** $L$ is a bounded lattice such that for all $a, b \in L$, there is a greatest element $x$ of $L$ with $a \land x \leq b$.
- This element is the **relative pseudocomplement** of $a$ with respect to $b$, and is denoted $a \rightarrow b$.
- A complete lattice is a Heyting algebra if and only if it satisfies (JID). Then,

$$a \rightarrow b = \bigvee \{c \mid a \land c \leq b\}$$

- Since $\mathcal{T}$ is closed under arbitrary unions and finite intersections, the complete lattice $(\mathcal{T}, \subseteq)$ satisfies (JID): for all $\mathcal{H} \subseteq \mathcal{T}$,

$$X \cap (\bigcup \mathcal{H}) = \bigcup\{X \cap Y \mid Y \in \mathcal{H}\}.$$  

Thus, every topology $\mathcal{T}$ determines a Heyting algebra.
Properties of rough approximations: quasiorders

An **Alexandrov topology** is a topology $\mathcal{T}$ that contains also all arbitrary intersections of its members. Let $\mathcal{T}$ be an Alexandrov topology $\mathcal{T}$ on $U$. Then, for each $X \subseteq U$, there exists the **smallest neighbourhood**

$$N_{\mathcal{T}}(X) = \bigcap \{ Y \in \mathcal{T} \mid X \subseteq Y \}.$$ 

In particular, the smallest neighbourhood of a point $x \in U$ is denoted by $N_{\mathcal{T}}(x)$. The family

$$\mathcal{B}_{\mathcal{T}} = \{ N_{\mathcal{T}}(x) \mid x \in U \}$$

is the **smallest base** of the Alexandrov topology $\mathcal{T}$. This means that every member $X$ of $\mathcal{T}$ can be expressed as a union of some (or none) elements of $\mathcal{B}_{\mathcal{T}}$, i.e. $X = \bigcup \{ N_{\mathcal{T}}(x) \mid x \in X \}$. In addition, $\mathcal{B}_{\mathcal{T}}$ is smallest such set.
Complete lattices of Alexandrov topologies

- Every Alexandrov topology $\mathcal{T}$ defines a complete lattice:
  \[
  \bigvee \mathcal{H} = \bigcup \mathcal{H} \quad \text{and} \quad \bigwedge \mathcal{H} = \bigcap \mathcal{H}
  \]
  for all $\mathcal{H} \subseteq \mathcal{T}$

- $(\mathcal{T}, \subseteq)$ is a distributive lattice

- In a complete lattice $L$, an element $a$ is **completely join-irreducible** if $a = \bigvee S$ implies $a \in S$ for every $S \subseteq L$.

- The set of completely join-irreducible elements of $\mathcal{T}$ is $\mathcal{J} = \{N(x) \mid x \in U\}$.

- The lattice $\mathcal{T}$ is **spatial**, i.e. each element can be given as a join of join-irreducibles.
We say that a closure operator is an **Alexandrov closure operator** if it satisfies for all $\mathcal{H} \subseteq \mathcal{T}$,

$$C(\bigcup \mathcal{H}) = \bigcup C(\mathcal{H})$$

As in case of topologies and Kuratowski closure operators, there is 1-to-1 correspondence between Alexandrov topologies and Alexandrov closure operators.
There is a 1-to-1 correspondence between quasiorders and Alexandrov topologies.

For a quasiorder $R$ on the set $U$, we can define an Alexandrov topology $\mathcal{T}_R$ on $U$ consisting of all “$R$-closed” subsets of $U$ with respect to the relation $R$:

$$\mathcal{T}_R = \{A \subseteq U \mid (\forall x, y \in U) \ x \in A \ \& \ x \ R \ y \implies y \in A\}$$
The set $R(x)$ is the smallest neighbourhood of the point $x$ in the Alexandrov topology $\mathcal{T}_R$.

Trivially, $y \in R(x)$ if and only if $x \mathcal{R} y$.

This hints how we may determine quasiorders by means of Alexandrov topologies.

If $\mathcal{T}$ is an Alexandrov topology, then the quasiorder $R_\mathcal{T}$ is defined by

\[ x \mathcal{R}_\mathcal{T} y \iff y \in N_\mathcal{T}(x). \]

The correspondences $R \mapsto \mathcal{T}_R$ and $\mathcal{T} \mapsto R_\mathcal{T}$ are 1-to-1.
Alexandrov topologies and quasiorders

- For a quasiorder \( R \), the rough approximations satisfy for all \( X \subseteq U \):

\[
X^{\wedge \nabla} = X^\wedge, \quad X^{\Delta \nabla} = X^\Delta, \quad X^{\nabla \Delta} = X^\nabla, \quad X^{\nabla \wedge} = X^\nabla.
\]

- The approximations determine two Alexandrov topologies:

\[
\wp(U)^\wedge = \wp(U)^\nabla \quad \text{and} \quad \wp(U)^\nabla = \wp(U)^\Delta
\]

- Note that \( \wp(U)^\nabla \) is the same as \( T_R \) above (\( R \)-closed subsets)

- Clearly, these topologies are dual, i.e. for all \( X \subseteq U \),

\[
X \in \wp(U)^\wedge \iff X^c \in \wp(U)^\nabla
\]
For the Alexandrov topology $\wp(U)^\uparrow = \wp(U)^\downarrow$:

(i) $\uparrow : \wp(U) \rightarrow \wp(U)$ is the smallest neighbourhood operator.

(ii) $\Delta : \wp(U) \rightarrow \wp(U)$ is the Alexandrov closure operator. Note that the family of closed sets for the topology $\wp(U)^\uparrow$ is $\wp(U)^\downarrow$; — and vice versa.

(iii) $\nabla : \wp(U) \rightarrow \wp(U)$ is the Alexandrov interior operator, i.e., it maps each set to the greatest open set contained into the set in question.

(iv) The set $\{\{x\}^\uparrow \mid x \in U\} = \{R^{-1}(x) \mid x \in U\}$ is the smallest base.
Similarly, for the topology $\wp(U) = \wp(U)$:

(i) $\triangle: \wp(U) \rightarrow \wp(U)$ is the smallest neighbourhood operator.

(ii) $\triangledown: \wp(U) \rightarrow \wp(U)$ is the Alexandrov closure operator.

(iii) $\triangledown: \wp(U) \rightarrow \wp(U)$ is the Alexandrov interior operator.

(iv) The set $\{ \{x\} \mid x \in U \} = \{ R(x) \mid x \in U \}$ is the smallest base.
Equivalence $E$ is a tolerance and a quasiorder.

For an equivalence $E$, $X^{\uparrow \downarrow} = X^{\uparrow}$ and $X^{\downarrow \uparrow} = X^{\downarrow}$.

Therefore, $\wp(U)^{\downarrow} = \wp(U)^{\uparrow}$ forms a completely distributive Boolean lattice — in fact, a complete field of sets.

The equivalence classes of $E$ are the atoms.