Knowledge Representation: A Rough Set Perspective

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Thanks!

- Dr. Davide Ciucci
- Università di Milano-Bicocca
- Speakers of the Summer School
- YOU: for attending my talks

- My wife Guili Liu and my daughter Kathleen Yao

Outline

- Philosophical considerations
- Conceptual formulation of rough set theory
- Multiview understanding of rough set approximations
- Moving beyond rough sets:
  - Orthopair and three-valued logic (Dr. Davide Ciucci)
  - Three-way decisions

Philosophical Considerations

- Questions:
  - Why study rough set theory?
  - What is rough set theory?
  - How to study rough set theory?
  - Why do I study rough set theory?

Philosophical Considerations

- Three types of studies of a theory or a field:
  - Content:
    What?
  - Methodology (*focus of my talks):
    How?
  - Scientometrics:
    Who?
    Major schools of thoughts, impact, historical perspectives, trends, latest developments ...

Philosophical Considerations in a rough-set context and a personal setting

- Content:
  - What is rough set theory?
  - How to interpret rough set theory?
  - What can I do?
- Methodology:
  - How to study rough set theory?
  - How can I do?
- Scientometrics:
  - What are the major schools of thought?
  - What are the main milestones?
  - Who can I collaborate with?
  - Who can I follow?

Edsger W. Dijkstra on Methodology

"The traditional curriculum teaches mathematical facts, i.e., it teaches existing theories, concepts, and methods that have withstood the test of time; on the doing of mathematics, however, it is almost totally silent."

0 Preface (Mathematical Methodology) (EWD 1059)

"This course is not about mathematical results but about doing mathematics..."

Honors course "Mathematical Methodology" (EWD1220), Spring 1996

Edsger W. Dijkstra on Methodology

"The first effect of teaching a methodology – rather than disseminating knowledge – is that of enhancing the capacities of the already capable, thus magnifying the difference in intelligence."

E.W. Dijkstra, The Humble Programmer, Communications of the ACM 15 (10), 859-866, 1972.

Methodology

- Multilevel: Proper division of difficulty/labor; focus on particular tasks at proper levels; ask the right questions at the right level.
- Multiview: Take different perspectives; add your unique points of view.
- Methods:
  - Breath-oriented (*first part of my talks) vs. depth-oriented (*subsequent parts of my talks) approaches
  - Constructive (*) vs. axiomatic approaches
  - Conceptual/definitional (*) vs. computational formulations

Leonardo da Vinci on granularity (big picture and details)

"Move some distance away, because then your work will appear smaller, and more of it can be taken in at a glance, and any lack of harmony or proportion ... will be more effortlessly seen."

"Every once and a while, go away and take a relaxing break, and then when you come back to your work, your judgment will be better – because remaining constantly at work will hinder your power of judgment."

Notable Traits of da Vinci: ability to look at the big picture while also paying careful attention to very specific and relevant details.

http://www.rodneyohebsion.com/leonardo-da-vinci.htm

Jerry R. Hobbs on granularity (multilevel)

"We look at the world under various grain sizes and abstract from it only those things that serve our present interest."

"Our ability to conceptualize the world at different granularities and to switch among these granularities is fundamental to our intelligence and flexibility."

"It enables us to map the complexities of the world around us into simpler theories that are computational tractable to reason in."

Hobbs, J.R., Granularity, Proceedings of the Ninth International Joint Conference on Artificial Intelligence, 432-435, 1985.

David Marr on multilevel

A full understanding of an information processing system involves explanations at various levels.

Computational theory	Representation and algorithm	Hardware implementation
What is the goal of the computation, why is it appropriate, and what is the logic of the strategy by which it can be carried out?	How can this computational theory be implemented? In particular, what is the representation for the input and output, and what is the algorithm for the transformation?	How can the representation and algorithm be realized physically?

Figure 1–4. The three levels at which any machine carrying out an information-processing task must be understood.

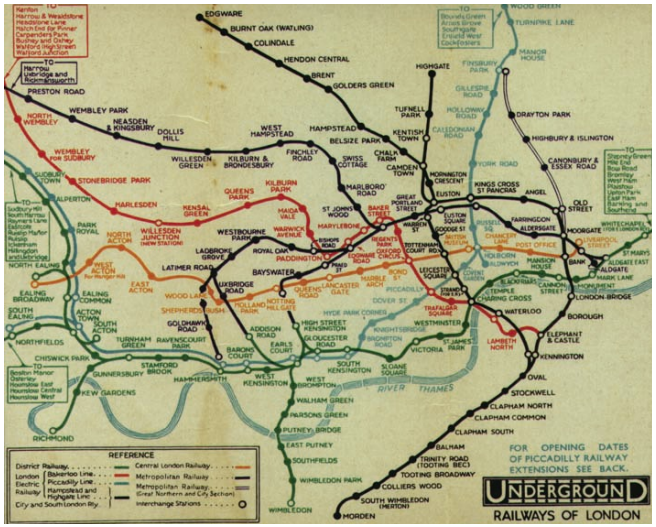
Marr, D., Vision: A Computational Investigation into the Human Representation and Processing of Visual Information, W.H. Freeman and Company, New York, 1982.

David Crystal on multilevel in language

- Enables us to see and state patterns of organization more clearly and succinctly.
  - Levels help us to simplify.
  - Levels help us to focus.
- Each level provides a distinct point of view.
- Each level can be studied based on its own terminology and techniques.

Crystal, D., How Language Works, Penguin Books, London, 2006.

An example of the effectiveness of the use of multilevel



Kramer, J., Is abstraction the key to computing? Communications of the ACM 50: 36-42, 2007.

Four-level understanding of multilelvel

We just presented a four-level understanding of multilevel from more general to more specific contexts:

- Level 1:
  Leonardo da Vinci: life and arts
- Level 2:
  Jerry R. Hobbs: problem-solving (artificial intelligence)
- Level 3:
  - David Marr: information processing systems
  - Cristal David: multiple levels in language
- Level 4: An example

Richard Feynman on different formulations

"Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. ... They are equivalent scientifically. ... But psychologically they are very different in two ways. ... psychologically they are different because they are completely unequivalent when you are trying to guess new laws.".

R. Feynman, The Character of Physical Law, The MIT Press, Cambridge, Massachusetts, 2001.

Basic Research Questions

- How to study rough set theory with respect to multilevel?
- How to study rough set theory with respect to multiview?

An example of study on methodology

- Constructive methods:
  intuitive, concrete, give a solution, may have difficulties in generalization.
- Axiomatic (algebraic) methods:
  abstract, show the existence of a solution, can be easily generalized.

Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, Information Sciences, Vol. 109, No. 1-4, pp. 21-47, 1998.

Another example of study on methodology

- Conceptual (definitional) formulation:
  Specify the meaning and interpretations of various notions.
- Computational formulation:
  Focus on methods/algorithms for constructing various notions.

Yiyu Yao, The two sides of the theory of rough sets, Knowledge-based Systems 80: 67-77, 2015.

An example of scientometrics studies

- Thirty years: 1982-2013
  Long-term, history, trends, major schools of thought
- Five years: 2008-2013
  Short-term (most recent), hot topics, new directions

JingTao Yao and Yan Zhang, A Scientometrics Study of Rough Sets in Three Decades, RSKT 2013, pp. 28-40.

An example of scientometrics studies

Table 4. Top cited 20 papers

	Paper	Total Citations	Average per Year	Main Results
1	Pawlak 1982 [29]	3694	115.44	Seminal paper, proposed RS
2	Ziarko 1993 [75]	659	31.38	Variable precision RS
3	Dubois+ 1990 [9]	565	23.54	Combining with fuzzy sets
4	Pawlak+ 2007 [36]	495	70.71	RS survey
5	Kryszkiewicz 1998 [23]	386	24.12	App - imcompete information table
6	Greco+ 2001 [15]	372	28.62	App - decision analysis
7	Pawlak+ 1995 [35]	352	18.53	Basic theory of RS
8	Slowinski+ 2000 [44]	323	23.07	Generalized RS
9	Pawlak+ 2007 [38]	315	45.00	RS survey
10	Mitra+ 2000 [28]	304	21.71	RS survey
11	Pawlak+ 2007 [37]	275	39.29	RS survey
12	Yao YY 1998 [61]	267	16.69	Research methods in RS
13	Kryszkiewicz 1999 [24]	249	16.60	App - imcompete information table
14	Yao YY 1998 [62]	236	14.75	Generalized RS using binary relation
15	Swiniarski+ 2003 [47]	231	21.00	App - feature selection
16	Pawlak 1998 [32]	231	14.40	App - data analysis
17	Yao YY 1996 [60]	219	12.17	Interpretation of RS
18	Wu+ 2003 [53]	205	18.64	Combining with fuzzy sets
19	Pawlak 2002 [34]	198	16.50	App - data analysis
20	Polkowski+ 1996 [39]	194	10.78	Basic theory of RS

JingTao Yao and Yan Zhang, A Scientometrics Study of Rough Sets in Three Decades, RSKT 2013, pp. 28-40.

An example of scientometrics studies

Table 1. Most prolific authors

Authors	Papers	Authors	Papers	Authors	Papers
Slowinski R	92	Hu QH	58	Grzymala-busse JW	43
Skowron A	91	Pal SK	57	Ramanna S	43
Yao YY	83	Miao DQ	55	Wang J	42
Wang GY	74	Chen DG	52	Polkowski L	41
Peters JF	72	Slezak D	52	Suraj Z	41
Wu WZ	70	Qian YH	48	Zhu W	40
Zhang WX	68	Li TR	47	Pawlak Z	39
Tsumoto S	67	Yu DR	46	Shi KQ	39
Greco S	66	Lin TY	44	Cheng CH	38
Liang JY	59	Ziarko W	44	Jensen R	37

Table 5. Impact authors

Authors	Cites	cts	Authors	Cites	cts	Authors	Cites	cts
Pawlak Z	5957	10	Kryszkiewicz M	749	3	Zhu W	474	4
Slowinski R	1689	8	Wu WZ	737	6	Grzymalabusse J	458	2
Skowron A	1510	5	Dubios D	641	2	Hu QH	407	3
Yao YY	1456	10	Prade H	641	2	Pal SK	392	4
Ziarko W	1088	3	Jensen R	548	5	Vanderpooten D	323	1
Zhang WX	798	6	Mi JS	545	4	Zopounidis C	312	2
Greco S	758	4	Shen Q	544	5	Yu DR	305	3
Matarazzo B	758	4	Mitra S	520	3	Hayashi Y	304	1

JingTao Yao and Yan Zhang, A Scientometrics Study of Rough Sets in Three Decades, RSKT 2013, pp. 28-40.

An example of scientometrics studies

Table 6. Top 20 cited papers in recent 5 years

	Paper	Total Citations	Average per Year	Main Results
1	Feng+ 2008 [10]	91	15.17	Soft sets
2	Yao YY+2008 [73]	81	13.50	Reduction in DTRS
3	Yao YY 2008 [66]	76	12.67	Probabilistic rough sets
4	Hu+ 2008 [19]	75	12.50	App - feature subset selection
5	Zhu 2009 [74]	69	13.80	Generalized RS
6	Hu+ 2008 [20]	65	10.83	App - neighborhood classifier
7	Jensen+ 2009 [21]	64	12.80	App - feature selection
8	Wu 2008 [52]	60	10.00	Attribute reduction
9	Qian+ 2010 [41]	55	13.75	Reduction accelerator
10	Wang+2008 [49]	52	8.67	App - rule induction
11	Thangavel+ 2009 [48]	48	9.60	Reduction (survey)
12	Liu 2008 [27]	48	8.00	Generalized RS
13	Qian+ 2008 [40]	48	8.00	Measures
14	Yang+ 2008 [55]	45	7.50	Dominance RS
15	Feng+ 2010 [11]	44	11.00	Soft sets
16	Yao YY 2010 [68]	41	10.25	Introduced three-way decision
17	Xiao+ 2009 [54]	41	8.20	App - forecasting
18	Bai+ 2010 [3]	38	9.50	Combining with grey system
19	Li+ 2008 [25]	38	6.33	App - prediction
20	Feng+ 2011 [12]	37	12.33	Soft sets

JingTao Yao and Yan Zhang, A Scientometrics Study of Rough Sets in Three Decades, RSKT 2013, pp. 28-40.

Conceptual vs. Computational Formulations: An Example

Suppose we have a list of raw scores $x_1, \ldots, x_n$, where n is the number of scores.

What is the meaning of the following formula?

$$s^2 = \frac{1}{n}\left(\sum_i^n x_i^2 - \frac{(\sum_i^n x_i)^2}{n}\right).$$

K. Guttmannova, A.L. Shields, J.C. Caruso, Promoting conceptual understanding of statistics: Definitional versus computational formulas, Teaching of Psychology 32 (2005) 251-253.

Conceptual vs. Computational Formulations: An Example

Suppose we have a list of raw scores $x_1, \ldots, x_n$, where n is the number of scores.

What is the meaning of the following pair of formulas?

$$m = \frac{\sum_i^n x_i}{n},$$
$$s^2 = \frac{\sum_i^n (x_i - m)^2}{n}.$$

m is the mean value and s^2 is the variance.

K. Guttmannova, A.L. Shields, J.C. Caruso, Promoting conceptual understanding of statistics: Definitional versus computational formulas, Teaching of Psychology 32 (2005) 251-253.

Conceptual vs. Computational Formulations: An Example

- Conceptual/definitional formulation:

$$\begin{aligned} m &= \frac{\sum_i^n x_i}{n}, \\ s^2 &= \frac{\sum_i^n (x_i - m)^2}{n}. \end{aligned}$$

- Computational formulation:

$$s^2 = \frac{1}{n}\left(\sum_i^n x_i^2 - \frac{(\sum_i^n x_i)^2}{n}\right).$$

Conceptual vs. Computational Formulations

- The two sides of the same coin.
- They mutually support and reinforce each other.
- Teaching of mathematics involves the following two aspects*:
  conceptual understanding vs.
  computational/procedural proficiency
  conceptual development vs.
  computational/procedural fluency
  conceptual knowledge and skills vs.
  procedural knowledge and skills.

* National Mathematics Advisory Panel, Foundations for Success: The Final Report of the National Mathematics Advisory Panel, 2008, http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

Conceptual Formulation of Rough Sets

- We revisit Pawlak's earlier works, in particular, those the motivated the introduction of rough set theory.
- We look at a knowledge representation and reasoning perspective of rough set theory.
- Computational formulations have been dominated the main stream rough set research.
- It is important to pay attention to conceptual formulations in rough set research.

Yiyu Yao, The two sides of the theory of rough sets, Knowledge-based Systems 80: 67-77, 2015.

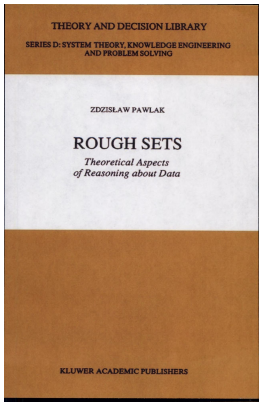
Earlier Works of Pawlak

- Z. Pawlak, Mathematical Foundations of Information Retrieval, Research Report CC PAS Report 101, Computation Center, Polish Academy of Sciences, 1973.
- V.W. Marek, Z. Pawlak, Information storage and retrieval systems: Mathematical foundations, Theoretical Computer Science 1: 331-354, 1976.
- Z. Pawlak, Information systems, theoretical foundations, Information Systems 6: 205-218, 1981.
- E. Orłowska and Z. Pawlak, Expressive Power of Knowledge Representation Systems, Research Report PAS 432, Institute of Computer Science, Polish Academy of Sciences, 1981.
- Z. Pawlak, Principles of knowledge representation, Bulletin of the Section of Logic 12(4): 194-199, 1983.

Following-up Studies

- V.W. Marek, M. Truszczyński, Contributions to the theory of rough sets, Fundamenta Informaticae 39: 389-409, 1999.
- Y.Y. Yao, A note on definability and approximations, LNCS Transactions on Rough Sets, VII, LNCS 4400, 274-282, 2007.
- V.W. Marek, Zdzisław Pawlak, databases and rough sets, in: A. Skowron, Z. Suraj (eds.), Rough Sets and Intelligent Systems, Springer, Berlin, 175-184, 2013.
- Y.Y. Yao, The two sides of the theory of rough sets, Knowledge-based Systems 80: 67-77, 2015.

The Seminal Book on rough sets by Pawlak



Pawlak, Z. Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, 1991.

- Reasoning about data
  - Discover knowledge from data
  - Realize the value of data
- A knowledge representation perspective of rough set theory.

Rough Sets as a Theory/Tool for Concept Analysis

- Concepts are basic units of human thought.
- Concepts are a vocabulary for human thinking, reasoning, and communication.
- Concept analysis is an important part of rough set analysis: rough-set concept analysis.

Yiyu Yao, Rough-set concept analysis: Interpreting RS-definable concepts based on ideas from formal concept analysis , Information Sciences 346-347: 442-462, 2016.

Intension-extension view of concepts

An important point of view of the Port-Royal Logic is an understanding of an idea or a concept jointly as a pair of an intension (or comprehension) and an extension (or denotation).

- The intension of a concept consists of all attributes or properties that are valid for all those objects to which the concept applies.
- The extension of a concept is the set of objects that are instances of the concept.

Arnauld, A., Nicole, P. Logic or the art of thinking. Cambridge University Press, Cambridge, 1996.

Buroker, J. Port royal logic. Stanford Encyclopedia of Philosophy.

http://plato.stanford.edu/entries/port-royal-logic/

Intension-extension view of concepts

- We can study intensions of concepts in a logic setting.
- We can study extensions of concepts in a set-theoretic setting.

- Rough-concept analysis is based on this intension-extension view of concepts and logic and set-theoretic settings for studying intensions and extensions.

A Conceptual Formulation of Rough Sets

- A knowledge representation system (KRS) (Also known as an information system, an information table, or a data table)
- Concepts in an information table
- Definable concepts (sets) in an information table
- Rough set approximations (i.e., approximations of undefinable sets by a pair of definable sets called the lower and upper approximations)

Information table

An information table T is defined by the following quadruple:

$$T = (OB, AT, \{V_a \mid a \in AT\}, \{I_a \mid a \in AT\}),$$

where OB is a finite nonempty set of objects called the universe, AT is a finite nonempty set of attributes, V_a is the domain of attribute a, and $I_a : U \longrightarrow V_a$ is an information function. We use $I_a(x)$ to denote the value of object x on attribute a.

Information table

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

Description language (AKA decision logic language)

A description language DL is recursively defined by using logic conjunction $\wedge$ and disjunction $\vee$, as well as parentheses, as follows:

$$(1) \quad \langle a, v \rangle \in DL, \text{ where } a \in AT, v \in V_a,$$
$$(2) \quad \text{if } p, q \in DL, \text{ then } (p) \wedge (q), (p) \vee (q) \in DL.$$

Formulas defined by (1) are called atomic formulas, descriptors, or attribute-value pairs. Formulas defined by (2) contain extra parentheses to indicate the order of computation.

Description language

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

$$\langle \text{Height}, \text{short} \rangle,$$
$$\langle \text{Height}, \text{short} \rangle \wedge \langle \text{Hair}, \text{blond} \rangle,$$
$$\langle \text{Height}, \text{short} \rangle \vee \langle \text{Hair}, \text{dark} \rangle$$

Satisfiability of a formula by an object

The satisfiability of formula p by object x, written as $x \models p$, is defined as follows:

$$\begin{aligned}
&\text{(i)} && x \models \langle a, v \rangle, \text{iff } I_a(x) = v, \\
&\text{(ii)} && x \models (p) \wedge (q), \text{iff } x \models p \text{ and } x \models q, \\
&\text{(iii)} && x \models (p) \vee (q), \text{iff } x \models p \text{ or } x \models q.
\end{aligned} \tag{1}$$

Satisfiability of a formula by an object

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

$$o_1 \models \langle \text{Height}, \text{short} \rangle, \quad o_2 \models \langle \text{Height}, \text{short} \rangle,$$
$$o_1 \models \langle \text{Height}, \text{short} \rangle \wedge \langle \text{Hair}, \text{blond} \rangle,$$
$$o_1 \models \langle \text{Height}, \text{short} \rangle \vee \langle \text{Hair}, \text{dark} \rangle,$$
$$o_5 \models \langle \text{Height}, \text{short} \rangle \vee \langle \text{Hair}, \text{dark} \rangle.$$

Meaning of a formula

For a formula $p \in DL$, the set of objects $\|p\| \subseteq OB$ defined by:

$$\|p\| = \{x \in OB \mid x \models p\} \tag{2}$$

is called the meaning set of formula p. The meaning set $\|p\|$ consists of all objects that satisfy p.

Meaning of a formula

We can establish the following linkage between logic and set operations:

$$
\begin{aligned}
&(\mathrm{m1}) \qquad \|\langle a, v\rangle\| = \{x \in OB \mid I_a(x) = v\},\\
&(\mathrm{m2}) \qquad \|(p) \wedge (q)\| = \|p\| \cap \|q\|,\\
&(\mathrm{m3}) \qquad \|(p) \vee (q)\| = \|p\| \cup \|q\|.
\end{aligned} \tag{3}
$$

That is, logic conjunction and disjunction are interpreted in terms of set intersection and union, respectively.

Meaning of a formula

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

$$\|\langle \text{Height}, \text{short}\rangle\| = \{o_1, o_2, o_3\},$$
$$\|\langle \text{Height}, \text{short}\rangle \wedge \langle \text{Hair}, \text{blond}\rangle\| = \{o_1, o_2, o_3\},$$
$$\|\langle \text{Height}, \text{short}\rangle \vee \langle \text{Hair}, \text{dark}\rangle\| = \{o_1, o_2, o_3, o_4, o_5, o_6\}.$$

Concepts and definable sets

For a formula $p \in DL$, we can construct a concept $(p, \|p\|)$, where p is the intension and $\|p\|$ is the extension.

Suppose a set of objects $X \subseteq OB$ is the extension of a concept. We call X a definable concept or a definable set if there exists a formula $p \in DL$ such that

$$X = \|p\|.$$

Otherwise, X is an undefinable set.

Concepts and definable sets

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

Concepts: $(\langle \text{Height}, \text{short} \rangle, \{o_1, o_2, o_3\})$,
$(\langle \text{Height}, \text{short} \rangle \wedge \langle \text{Hair}, \text{blond} \rangle, \{o_1, o_2, o_3\})$,
$(\langle \text{Height}, \text{short} \rangle \vee \langle \text{Hair}, \text{dark} \rangle, \{o_1, o_2, o_3, o_4, o_5, o_6\})$.
Definable sets: $\{o_1, o_2, o_3\}$, $\{o_1, o_2, o_3, o_4, o_5, o_6\})$.
Undefinable sets: $\{o_1, o_2\}$, $\{o_4, o_5, o_7\}$.

The family of definable sets

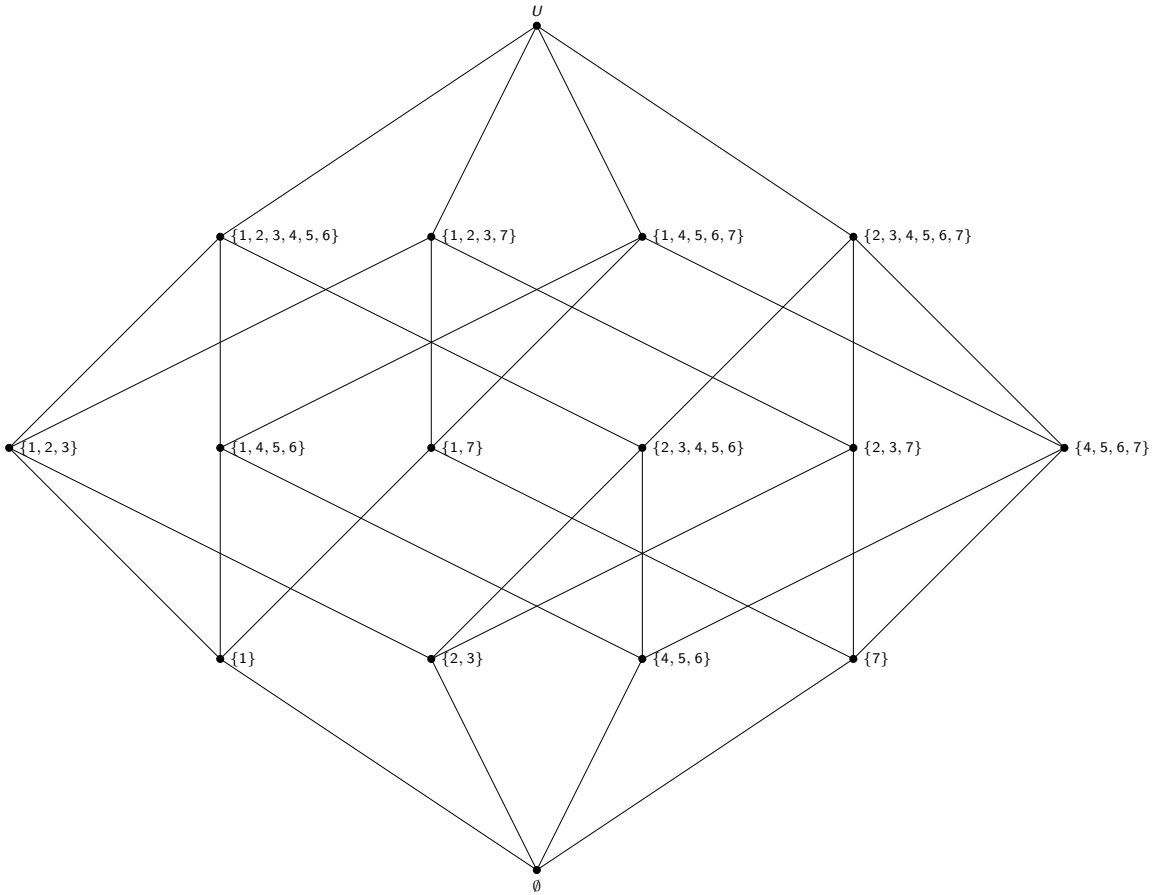
In rough set analysis, we are interested in the family of all definable set $\mathrm{DEF}(T)$.
$\mathrm{DEF}(T)$ is an atomic Boolean algebra.

The family of definable sets

Object	Height	Hair	Eyes
o_1	short	blond	blue
o_2	short	blond	brown
o_3	short	blond	brown
o_4	tall	dark	blue
o_5	tall	dark	blue
o_6	tall	dark	blue
o_7	tall	red	blue

$$\begin{aligned}\mathrm{DEF}(T) = \{&\emptyset, \{o_1\}, \{o_2, o_3\}, \{o_4, o_5, o_6\}, \{o_7\},\\ &\{o_1, o_2, o_3\}, \{o_1, o_4, o_5, o_6\}, \{o_1, o_7\},\\ &\{o_2, o_3, o_4, o_5, o_6\}, \{o_2, o_3, o_7\}, \{o_4, o_5, o_6, o_7\}\\ &\{o_1, o_2, o_3, o_4, o_5, o_6\}, \{o_1, o_2, o_3, o_7\}\\ &\{o_1, o_4, o_5, o_6, o_7\}, \{o_2, o_3, o_4, o_5, o_6, o_7\}, \ U\}.\end{aligned}$$

The family of definable sets



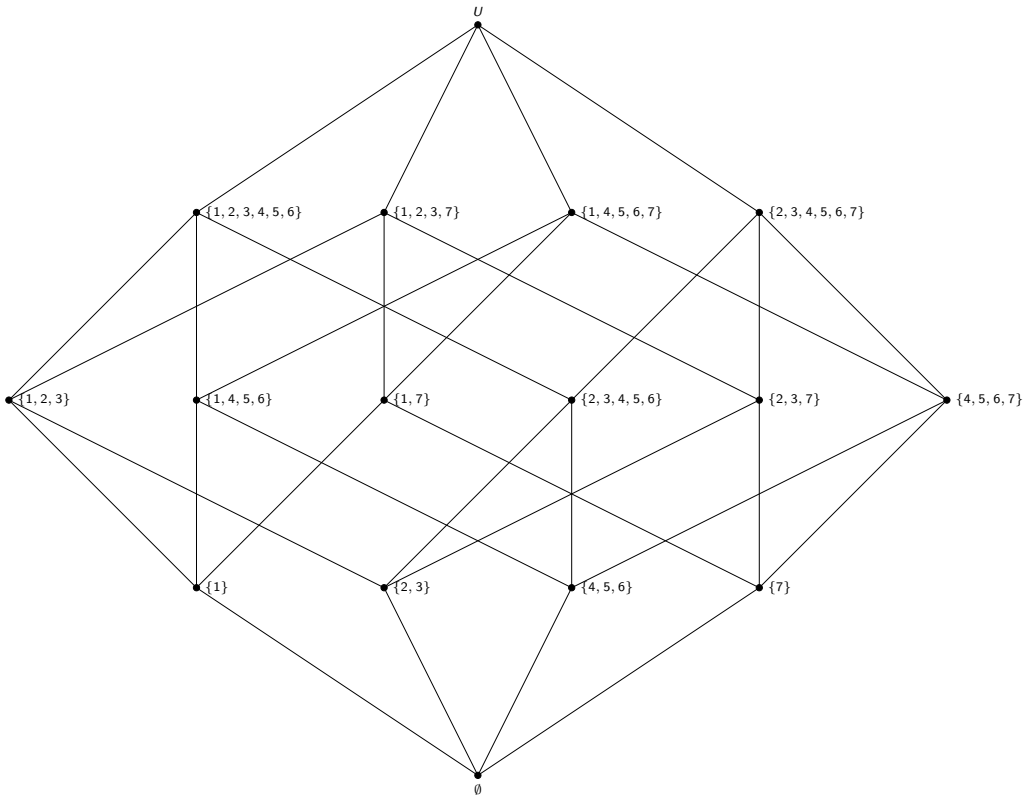
Rough set approximations

For a subset $X \subseteq OB$, the lower and upper approximations of X are defined by the following pair of definable sets,

$$\begin{aligned} \underline{apr}(X) &= \text{the greatest definable set in DEF}(T) \text{ contained by } X, \\ \overline{apr}(X) &= \text{the least definable set in DEF}(T) \text{ containing } X, \end{aligned}$$

where the greatest and the least definable sets are defined with respect to the set-inclusion relation $\subseteq$. We call the pair $(\underline{apr}(X), \overline{apr}(X))$ the rough set induced by X.

Rough set approximations



$$\underline{apr}(\{o_4, o_5, o_7\}) = \{o_7\},$$
$$\overline{apr}(\{o_4, o_5, o_7\}) = \{o_4, o_5, o_6, o_7\}.$$

Summary of conceptual formulation

- Definablity is a primitive notion.
- Rough set approximations are needed for approximating an undefinable set by a pair of definable sets.
- The meaning of approximations is clear.
- The construction of approximations directly from the conceptual formulation is difficult or computationally expensive.
- We need to search for a computational formulation. This may explain why computational formulations have been dominated main stream rough set research.

Richard Feynman on different formulations

"Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. ... They are equivalent scientifically. ... But psychologically they are very different in two ways. ... psychologically they are different because they are completely unequivalent when you are trying to guess new laws.".

R. Feynman, The Character of Physical Law, The MIT Press, Cambridge, Massachusetts, 2001.

A Computational Formulation of Rough Set Approximations

- Information table
- Equivalence relation induced by the set of attributes and the associated partition of the universe.
- Atomic Boolean algebra with the partition as its set of atoms.
- Rough set approximations.

Equivalence relation

In an information table, we define an equivalence relation E_{AT} as follows:

$$xE_{AT}y \Longleftrightarrow \forall a \in AT(I_a(x) = I_a(y)).$$

The equivalence class containing x is given by:

$$[x]_{AT} = \{y \in OB \mid xE_{AT}y\}.$$

The partition of E_{AT} is written as

$$OB/E_{AT} = \{[x]_{AT} \mid x \in OB\}.$$

For simplicity, we will drop subscript AT.

Atomic Boolean algebra

The atomic Boolean algebra with OB/E is given by $\mathbb{B}(OB/E)$.

Main results: connecting the two formulations

$$\mathrm{DEF}(T) = \mathbb{B}(OB/E).$$

Rough set approximations

$$\begin{aligned} \underline{apr}(X) &= \text{the greatest set in } \mathbb{B}(OB/E) \text{ contained by } X, \\ \overline{apr}(X) &= \text{the least set in } \mathbb{B}(OB/E) \text{ containing } X, \end{aligned}$$

Summary of conceptual formulation

- The motivations introducing rough set approximations are no longer explicit.
- It will be computationally easy to construct approximations (to be shown later).
- Computational formulations are relatively easy to follow (due to a lack of consideration of semantics).
- Computational may have negatively effected rough set research.

Multiview Understanding of Rough Set Approximations

- Three basic definitions of rough set approximations and their implications:
  - Element-based definition
  - Granule-based definition
  - Subsystem-based definition
- Three different representations of approximations:
  - A pair of lower and upper approximations
  - Orthopair: A pair of disjoint sets
  - Tri-partition: three pair-wise disjoint sets
- Two forms for definition rough set approximations:
  - Based on set inclusion and set intersection
  - Based only on set inclusion

Moving Beyond Rough Sets

Three-way Decisions (3WD)

Introduction Remarks

Three eras of computing:

Image 1

The Tabulating Era (1900s–1940s)

The Programming Era (1950s–present)

The Cognitive Era (2011–)

Three eras: Past, Present, Future

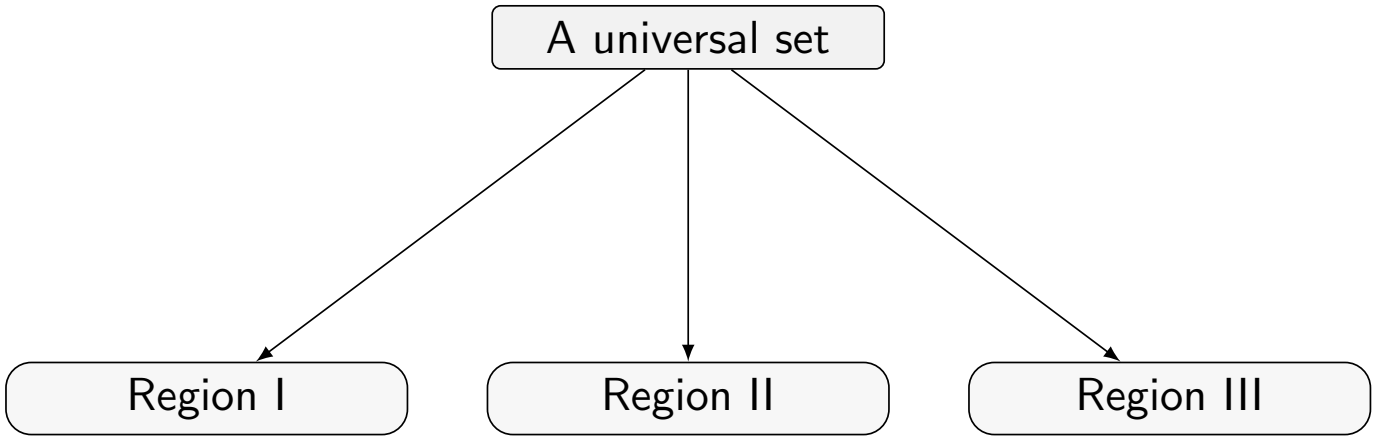
John E. Kelly III, Computing, cognition and the future of knowing, How humans and machines are forging a new age of understanding, 2015.

Introduction Remarks

- Three-Way Decisions (3WD) are a class of human ways to problem solving and information processing.
- Basic questions:
  - Cognitive basis of 3WD.
  - Cognitive advantages and benefits of 3WD.
  - Human-machine systems based on 3WD.

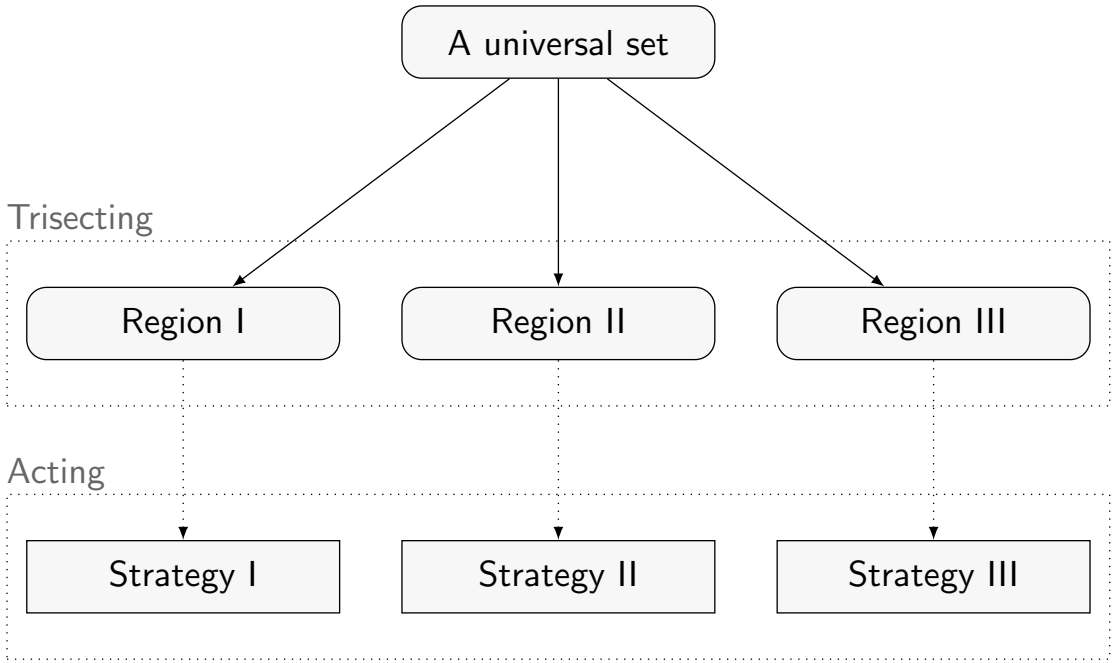
A Trisecting-and-acting (三分而治) Framework of 3WD

We divide the whole into three parts. For example, we divide a universal set into three regions:



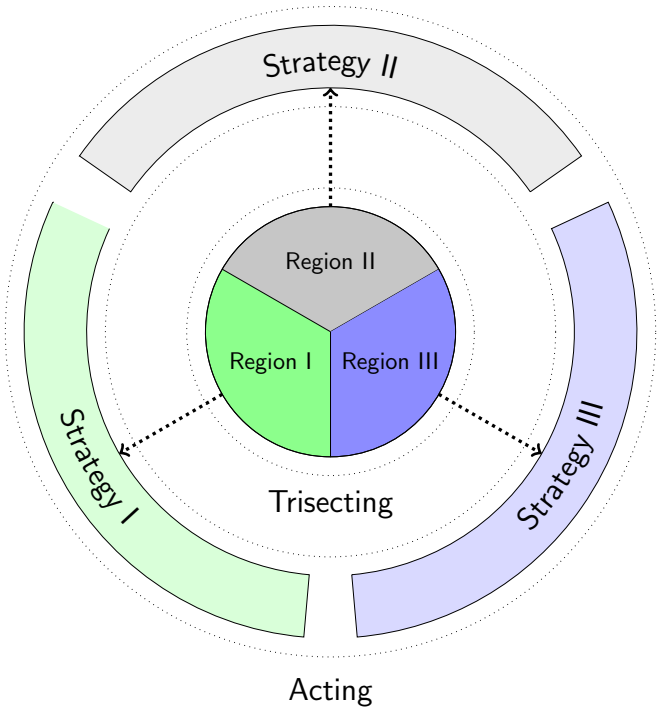
A Trisecting-and-acting (三分而治) Framework of 3WD

We use different strategies to process three regions:



姚一豫, 于 洪, 三支决策概述, in: 于洪, 王国胤, 李天瑞, 梁吉业, 苗夺谦, 姚一豫, (编著) 《三支决策：复杂问题求解方法与实践》 科学出版社, 北京, pp. 1-19, 2015.

A Trisecting-and-acting (三分而治) Framework of 3WD



Yao, Y.Y., Rough sets and three-way decisions. RSCTC 2015. LNAI 9436, pp. 62–73, 2015.

Two Basic Components of Three-Way Decisions

- Trisecting: We divide a universal set into three regions.
- Acting: We use different strategies to process three regions

A Broad Meaning of Three-Way Decisions

We may replace "decisions" in "three-way decisions" by other words to have specific interpretations/models:

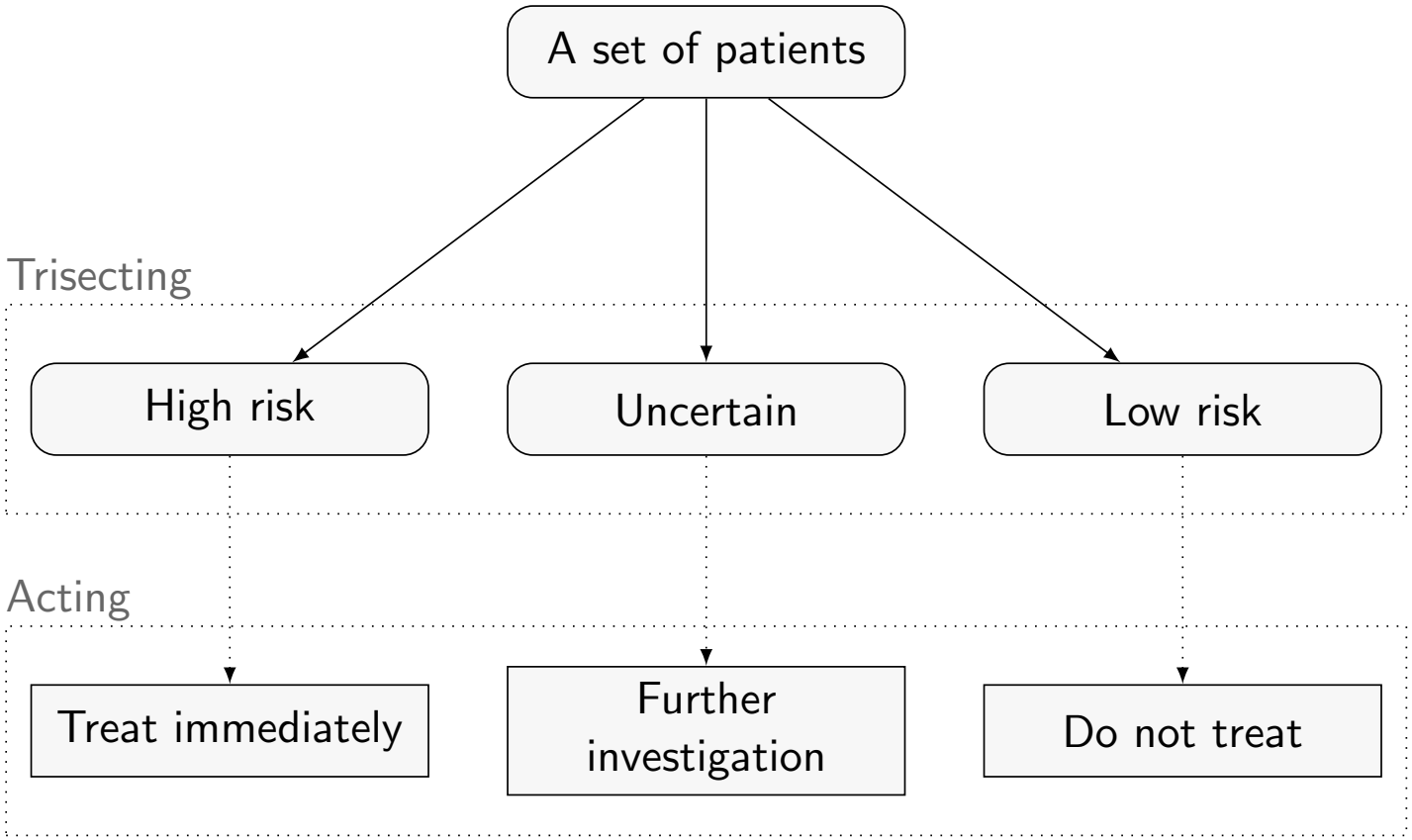
- three-way computing
- three-way processing
- three-way classification
- three-way analysis
- three-way clustering
- three-way recommendation
- tri-level analysis

Y.Y. Yao, Three-way decisions and cognitive computation, 2016

Research Questions

- Is the proposed theory/model reasonable?
- Is the proposed theory/model flexible and general enough?
- Is the proposed theory/model applicable to real-world problems?
- ...
- **Am I on the right track?**

Medical Decisions



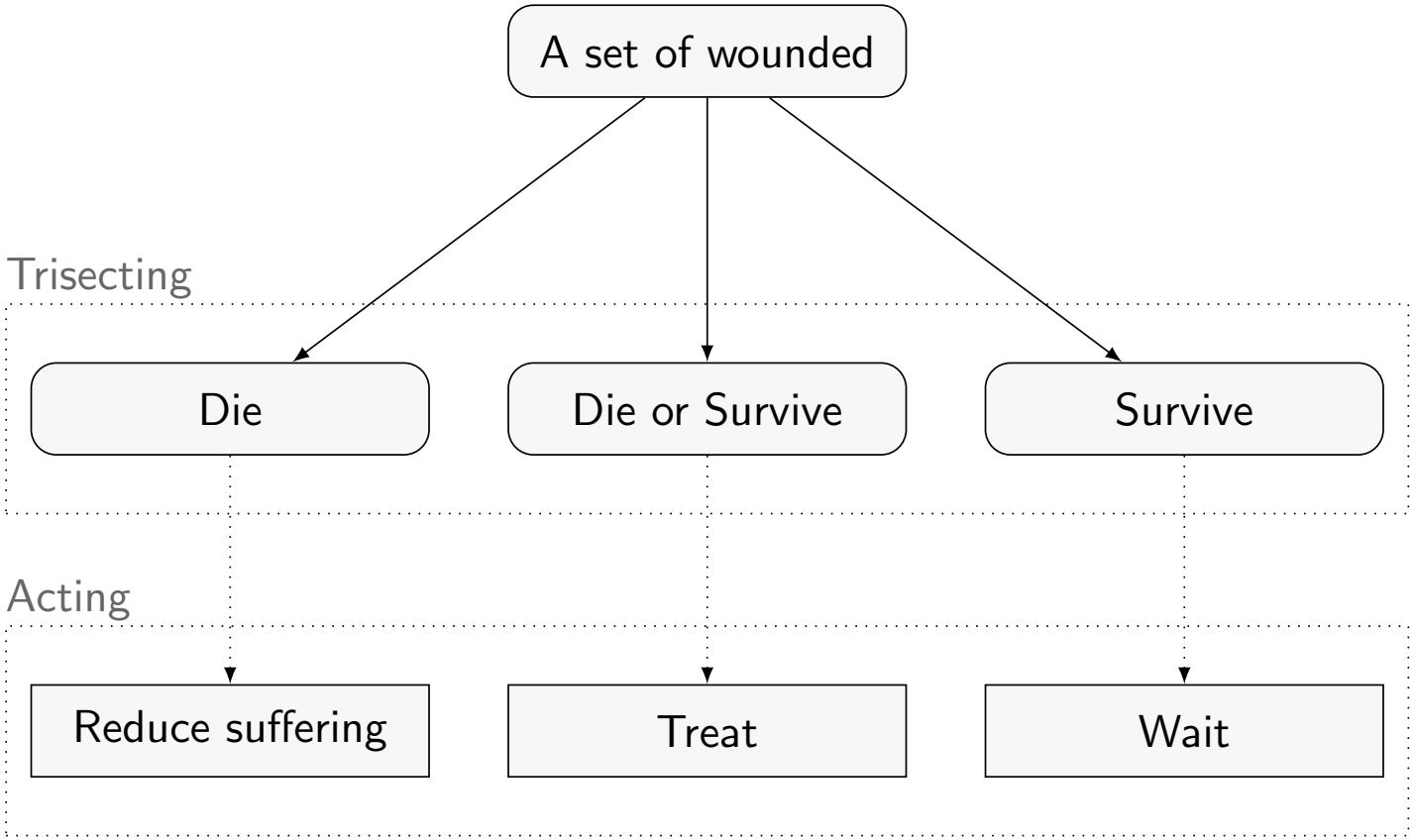
Pauker, S.G., Kassirer, J.P.: The threshold approach to clinical decision making. The New England Journal of Medicine 302, 1109-1117 (1980)

Triage

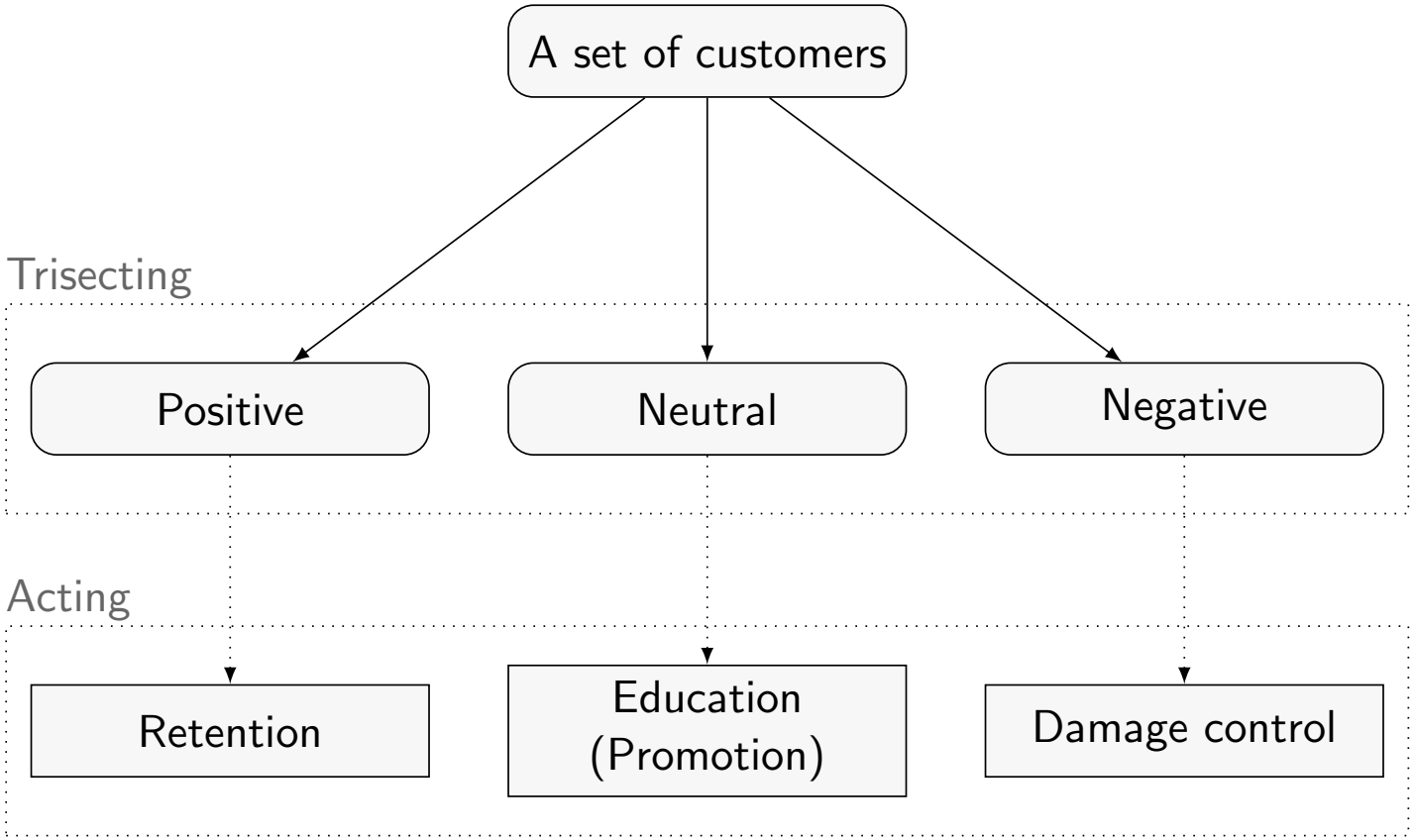
- Triage is the process of determining the priority of patients' treatments based on the severity of their condition.
- Those responsible for the removal of the wounded from a battlefield or their care afterwards would divide the victims into three categories:
  - Those who are likely to live, regardless of what care they receive;
  - Those who are likely to die, regardless of what care they receive;
  - Those for whom immediate care might make a positive difference in outcome.

Wikipedia, Triage, http://en.wikipedia.org/wiki/Triage

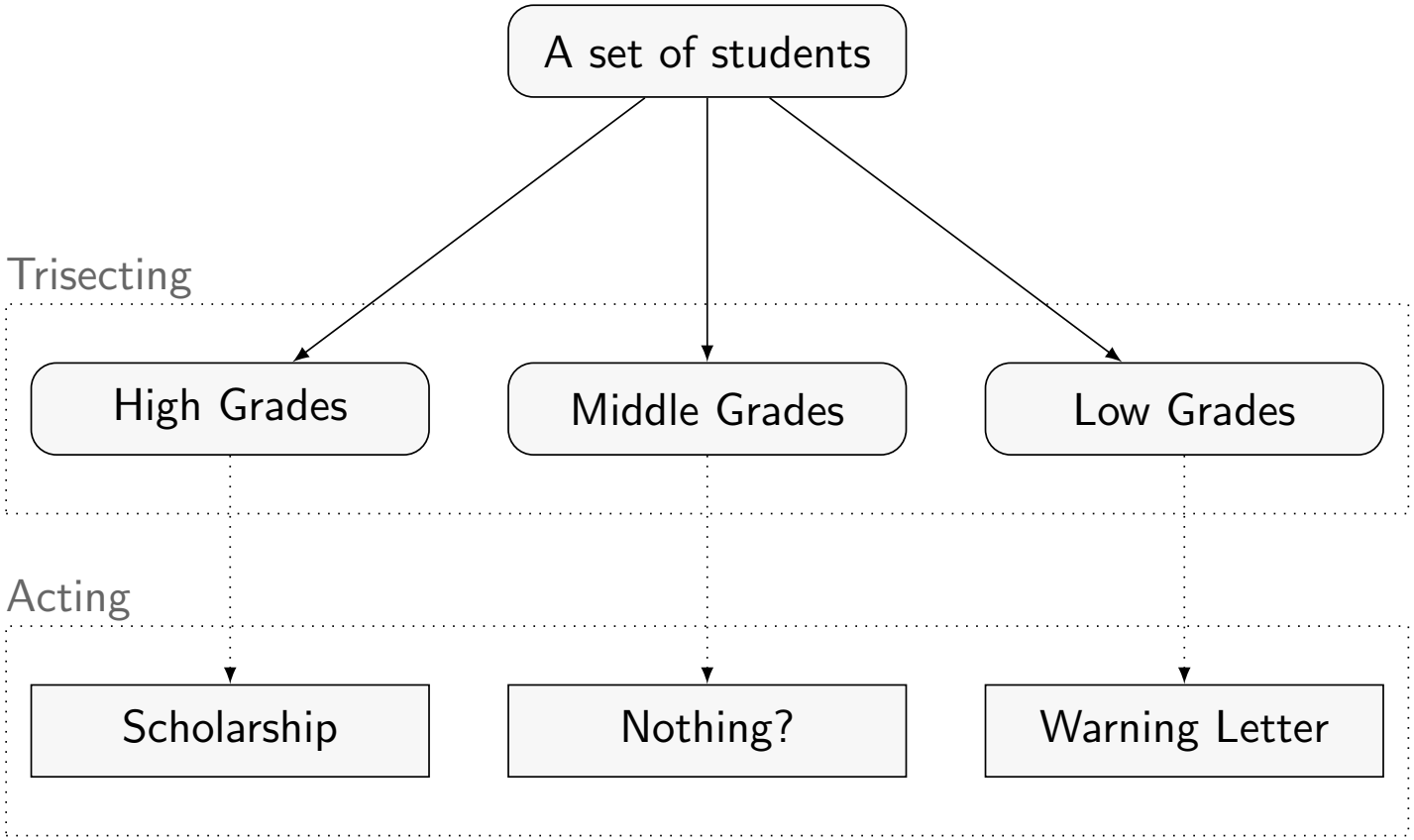
Triage


A set of wounded
Trisecting
Die
Die or Survive
Survive
Acting
Reduce suffering
Treat
Wait

Customer Relation Management



Student Management



Three-way decisions are everywhere: Temporal

- yesterday, today, tomorrow
- past, present, future

Three-way decisions are everywhere: Spatial

- top, middle, bottom,
- front, middle, back
- left, center, right

Three-way decisions are everywhere: Size and volume

- long, medium, short
- high, medium, low
- large, medium, small

Three-way decisions are everywhere: Attitude

- positive, neutral, negative
- accept, non-commitment, reject

Three-way decisions are everywhere: Evaluation

- yes/right, maybe, no/wrong
- upper/top, middle, lower/bottom
- good, so-so, bad

Research Questions

- Why the theory is meaningful? (Basis)
- How to build the theory? (Formulation)
- How to apply the theory? (Utility)

Cognitive Basis

- Organization and categorization for simplicity.
- Limited human information processing capacity.
- Evolutionary needs for fast decision-making.

Y.Y. Yao, Three-way decisions and cognitive computing, Cognitive Computation 8, 543-554, 2016.

Organization and Categorization

- Humans tend to organize for the sake of simplicity.
- Categorization is essential to mental life.
- Possible results of such organizations are some types of structures.

- **In three-way decisions, we have a tri-partition**.

S. Pinker, How the Mind Works, WW Norton & Company, New York, 1997.

Why 3?

- Three is a magical number.
- Three is a widely used number in human perception, reasoning, and decision-making.

- **The choice of three is appropriate**.

Limited Human Information Processing Capacity

- G.A. Millller (1956), The magical number seven, plus or minus two: Some limits on our capacity for processing information, Psychological Review 101, 343-352.
- N. Cowan (2001), The magical number 4 in short-term memory: A reconsideration of mental storage capacity, Behavioral and Brain Sciences 24, 87-114.

- **The choice of three-way decisions is 3!**

Warfield, 1988

Given a tri-partition (A, B, C), we consider

- single regions: A, B, C,
- combinations of two regions: (A, B), (A, C), (B, C)
- combination of three regions: (A, B, C)
- The total number is $3 + 3 + 1 = 7$, which is exactly seven.

If we use four parts, we would have a total of $4 + 6 + 4 + 1 = 15$, which is far more beyond the capacity as suggested by seven.

Warfield JN. The magical number three – plus or minus zero. Cybernetics and Systems 1988; 19: 339-358.

The magical number three in human cognition

- Three is the first number with a "beginning, middle, and end."
- This structure determined by dividing into three is commonly used, for example, in speeches, writings, planning, etc.

Marketing

Versioning strategies often follow a qualitatively three-part "good-better-best" progression, with increasing number of features and increasing pricing.

Smith T. Pricing strategy: setting price levels, managing price discounts, and establishing price structure. Mason, Ohio: South-Western Cengage Learning; 2012.

Argument and persuasion

The optimal number of positive claims is three in order to produce the most positive impression of a product or a service.

Shu SB, Carlson KA. When three charms but four alarms: identifying the optimal number of claims in persuasion settings. Journal of Marketing 2014; 78: 127-139.

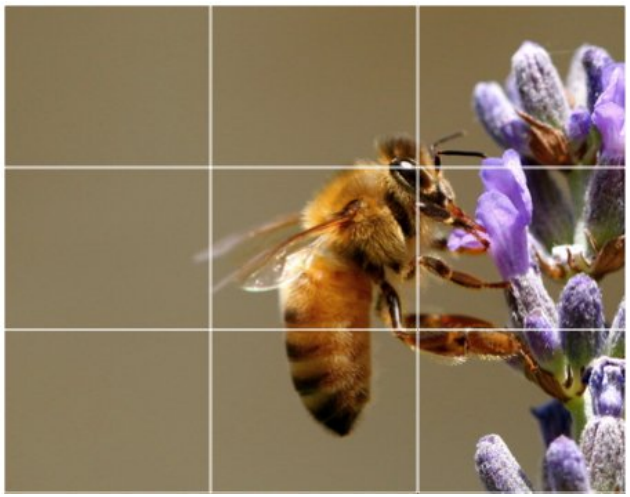
Speeches

Lists of three things (e.g., three words, three phrases, or three sentences) are powerful speech patterns commonly used by great speakers.

Clayton M. Brilliant influence: what the most influential people know, do and say. New York: Prentice Hall; 2011.

Three empirical laws

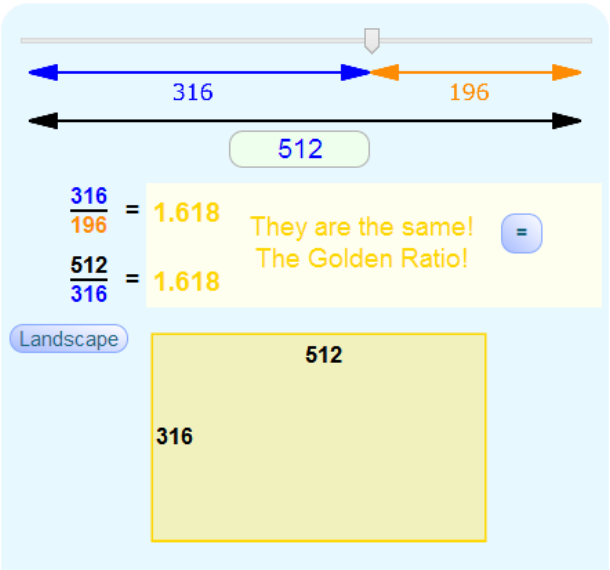
The rule of thirds: to divide a medium into thirds both horizontally and vertically, (0.00-0.33, 0.33-0.67, 0.67-1.00).



http://digital-photography-school.com/rule-of-thirds/

Three empirical laws

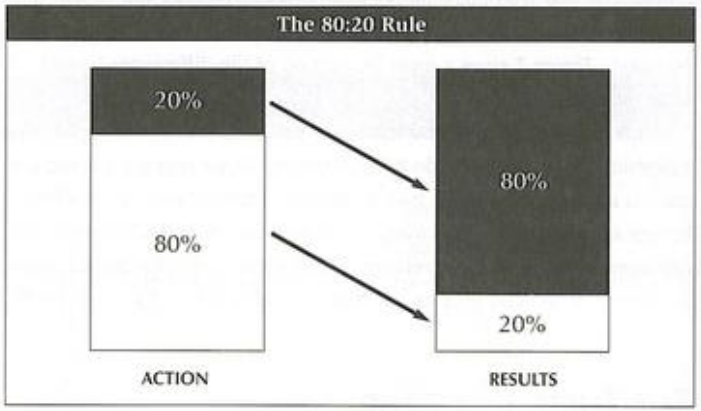
Golden ratio: (0.00-0.382, 0.382-0.618, 0.618-1.00)



http://www.zcool.com.cn/article/ZMTcyNTI0.html

Three empirical laws

Pareto Principle or 20/80 Laws: (20%, 60%, 20%)



http://musicianguide.cn/8020-rule-in-music-social-media-marketing?utm_source=weixin&utm_medium=mgarticle&utm_campaign=socialmedia

Evolutionary Needs for Fast Decision-Making

Two modes of thinking, two systems in the mind:

- System 1 operates automatically and quickly, with little or no effort and no sense of voluntary control.
- System 2 allocates attention to the effortful mental activities that demand it, including complex computations.

- **Some regions are suitable for system 1 and some for system 2**.

D. Kahneman, Thinking, Fast and Slow, Anchor Canada, 2013.

Three-Way Decisions: Cognitive Advantages

- Three regions correspond to two extreme points (i.e., the two polarities) and a middle point.
- Humans can easily determine and process objects that are instances of the two extreme cases.
- Three presents a small number of regions that are easy to manage.

Three-Way Decisions: Benefits

- Trade-off between accuracy and cost.
- Trade-off between effectiveness and efficiency.
- Trade-off between benefits of the majority and the benefits of a few.
- Trade-off between risk of immediately decisions and deferred decisions.

Research Questions

- How to determine the three regions?
- How to effectively process each region?

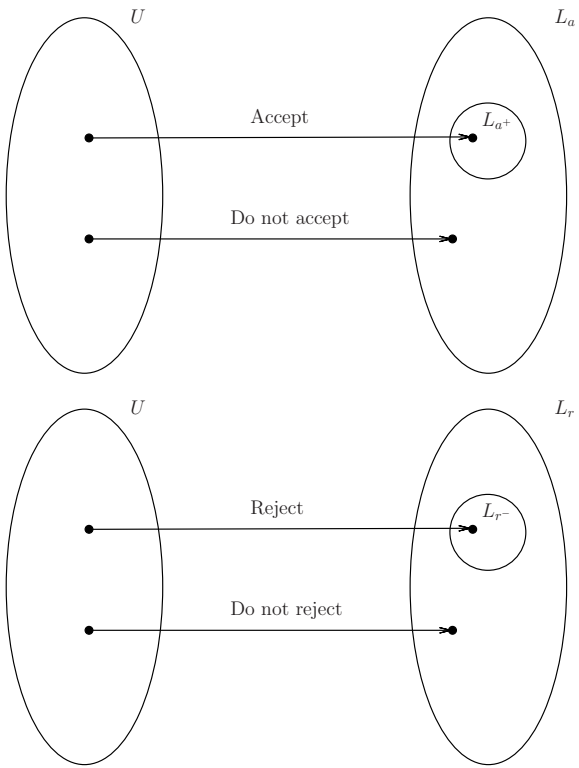
- **A theory of three-way decisions can be formally developed**.

Y.Y. Yao, An Outline of a Theory of Three-way Decisions, RSCTC 2012, LNCS (LNAI) 7413, pp. 1-17, 2012.

Basic Ideas

- Build evaluation functions for dividing objects into three regions:
  - Determine and interpret the values of evaluation functions for three-way decisions.
  - Determine three regions based on evaluation status values.
- Utilize the three regions, that is, process the three regions by designing best strategies.

Model 1: A Pair of Poset-based Evaluations



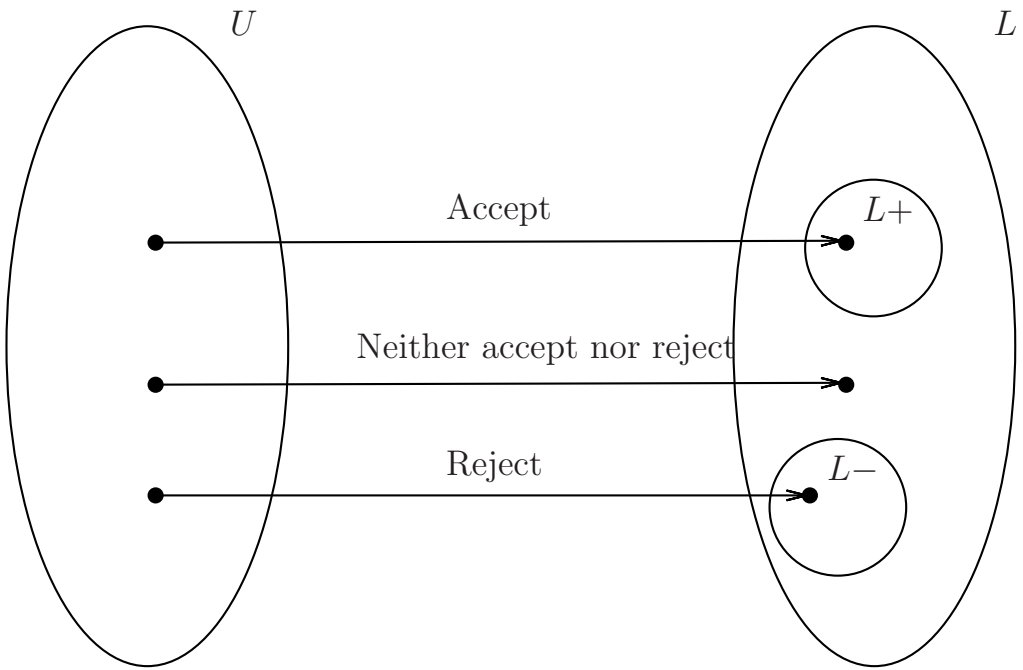
2

Model 1: A Pair of Poset-based Evaluations

The three regions are defined by:

$$
\begin{aligned}
\mathrm{L}_{(L_a^+, L_r^-)}(v_a, v_r) &= \{x \in U \mid v_a(x) \in L_a^+ \wedge v_r(x) \notin L_r^-\}, \\
\mathrm{R}_{(L_a^+, L_r^-)}(v_a, v_r) &= \{x \in U \mid v_a(x) \notin L_a^+ \wedge v_r(x) \in L_r^-\}, \\
\mathrm{M}_{(L_a^+, L_r^-)}(v_a, v_r) &= (\mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r) \cup \mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r))^c \\
&= \{x \in U \mid (v_a(x) \notin L_a^+ \wedge v_r(x) \notin L_r^-) \vee \\
&\qquad (v_a(x) \in L_a^+ \wedge v_r(x) \in L_r^-)\}.
\end{aligned}
$$

Model 2: One Poset-based Evaluation

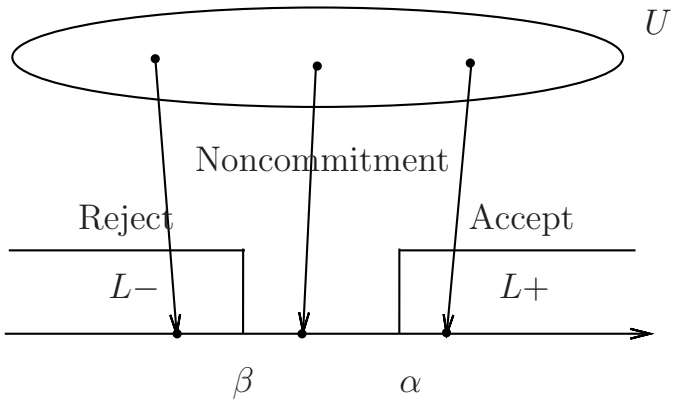


Model 2: One Poset-based Evaluation

The three regions are defined by:

$$\begin{aligned}
\mathrm{L}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \in L^+\},\\
\mathrm{R}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \in L^-\},\\
\mathrm{M}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \notin L^+ \wedge v(x) \notin L^-\}.
\end{aligned}$$

Model 3: A Totally Ordered Set based Evaluation



Model 3: A Totally Ordered Set based Evaluation

The three regions are defined by:

$$\begin{aligned}
\mathrm{L}_{(\alpha,\beta)}(v) &= \{x \in U \mid v(x) \succeq \alpha\},\\
\mathrm{R}_{(\alpha,\beta)}(v) &= \{x \in U \mid v(x) \preceq \beta\},\\
\mathrm{M}_{(\alpha,\beta)}(v) &= \{x \in U \mid \beta \prec v(x) \prec \alpha\}.
\end{aligned}$$

Construction of Evaluations (An Example)

Suppose $C = \{c_1, c_2, \ldots, c_m\}$ are a set of m criteria. Suppose $v_{c_i} : U \longrightarrow \Re$ denotes an evaluation based on criterion v_i, $1 \leq i \leq m$. An overall evaluation function $v : U \longrightarrow \Re$ may be simply defined by a linear combination of individual evaluations:

$$v(x) = w_1 v_{c_1}(x) + w_2 v_{c_2}(x) + \ldots + w_m v_{c_m}(x).$$

Connection to multi-criteria decision making (MCDM)

Determination of Designated Sets of Values (An Example)

Let $R_L(\alpha, \beta)$, $R_R(\alpha, \beta)$ and $R_M(\alpha, \beta)$ denote the risks of the three regions, respectively. It is reasonable to require that the sets of designated values are chosen to minimize the following overall risks:

$$R(\alpha, \beta) = aR_L(\alpha, \beta) + bR_M(\alpha, \beta) + cR_R(\alpha, \beta).$$

$$\arg \min_{(\alpha,\beta)} R(\alpha, \beta).$$

Connection to multi-objective decision making (MODM)

Special Models of Three-Way Decisions

- Rough Sets
- Interval Sets
- Three-way Approximations of Fuzzy Sets
- Shadowed Sets
- Three-way Bayesian Confirmation
- Orthopairs (pairs of disjoint sets, set pairs)

Examples of 3WD research

- Theories and Models of 3WD
- Cluster Analysis
- Classification
- Email Spam Filtering
- Government Decision-Making
- Decision-Support Systems
- Sentiment Analysis
- Abnormality Detection

Three Chinese Books

- Hong, Y., Wang, G.Y., Li, T.R., Liang, J.Y., Miao, D.Q., Yao, Y.Y. (Eds.) Three-way Decisions: Methods and Practice of Complex Problem Solving, Science Press, Beijing, 2015

  于洪, 王国胤, 李天瑞, 梁吉业, 苗夺谦, 姚一豫. 三支决策: 复杂问题求解方法与实践, 科学出版社, 北京, 2015

- Liu, D., Li, T.R., Miao, D.Q., Wang, G.Y., Liang, J.Y. (Eds.), Three-way Decisions and Granular Computing, Science Press, Beijing, China, 2013.

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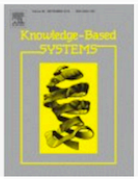
  贾修一，商琳，周献中，梁吉业，苗夺谦，王国胤，李天瑞，张燕平，三支决策理论与应用，南京：南京大学出版社，2012.

Special Issue

- Special issue on "three-way decisions and granular computing" in Knowledge-based System (Guest editors: Hamido Fujita, Tianrui Li, Yiyu Yao)
- Homepage in Google Scholar: `https://scholar.google.ca/citations?hl=en&user=btM_CLsAAAAJ&view_op=list_works`

Special Issue



3WD on KBS

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Title 1–20	Cited by	Year
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Generalized attribute reduct in rough set theory X Jia, L Shang, B Zhou, Y Yao Knowledge-Based Systems 91, 204-218	18	2016
A novel three-way decision model based on incomplete information system D Liu, D Liang, C Wang Knowledge-Based Systems 91, 32-45	16	2016
Three-way recommender systems based on random forests HR Zhang, F Min Knowledge-Based Systems 91, 275-286	14	2016
Sequential three-way decision and granulation for cost-sensitive face recognition H Li, L Zhang, B Huang, X Zhou Knowledge-Based Systems 91, 241-251	13	2016
Decision-theoretic rough sets under dynamic granulation Y Sang, J Liang, Y Qian Knowledge-Based Systems 91, 84-92	12	2016
A comparative study of multigranulation rough sets and concept lattices via rule acquisition J Li, Y Ren, C Mei, Y Qian, X Yang Knowledge-Based Systems 91, 152-164	9	2016
Proximal three-way decisions: theory and applications in social networks JF Peters, S Ramanna Knowledge-Based Systems 91, 4-15	8	2016
Ranking interval sets based on inclusion measures and applications to three-way decisions HY Zhang, SY Yang, JM Ma Knowledge-Based Systems 91, 62-70	7	2016
Three-way decision spaces based on partially ordered sets and three-way decisions based on hesitant fuzzy sets BQ Hu	3	2016

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- Ning Zhong, Jiajin Huang: Granular Structures Induced by Interval Sets and Rough Sets. RSKT 2015, pp. 49-60.
- Yiyu Yao: Rough Sets and Three-Way Decisions. RSKT 2015, pp. 62-73.

2016大数据决策高峰论坛 (2016 Forum of Big Data and Decision Making

- 李天瑞, 面向大数据的动态知识发现与三支决策.
- 苗夺谦, 不确定性与三支决策.
- 于 洪, 三支决策聚类.
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Three-Way Cluster Analysis

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- S.J. Liao, Q.X., Zhu, F. Min, Cost-sensitive attribute reduction in decision-theoretic Rough set models, Mathematical Problems in Engineering 1-14, 2014

Email Spam Filtering

- X.Y. Jia, L. Shang, Three-way decisions versus two-way decisions on filtering spam email, Transactions on Rough Sets XVIII (2014), LNCS 8449 69-91.
- B. Zhou, Y.Y. Yao, J.G. Luo, Cost-sensitive three-way email spam filtering, Journal of Intelligent Information Systems 42 (2014) 19-45.
- J.L. Li, X.F. Deng, Y.Y. Yao, Multistage email spam filtering based on three-way decisions, RSKT 2013, 313-324, 2013.

Government Decision-Making

- D. Liu, T.R. Li, D.C. Liang, Three-way government decision analysis with decision-theoretic rough sets. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 20 (2012) 119-132.
- D. Liu, Yao, Y.Y., T.R. Li, Three-way investment decisions with decision-theoretic rough sets International Journal of Computational Intelligence Systems 4 (2011) 66-74.

Three-way Recommendation

- H.R. Zhang, F. Min, B. Shi, Regression-based three-way recommendation, Information Sciences, 2016.
- H.R. Zhang, F. Min, Three-way recommender systems based on random forests, Knowledge-Based Systems 91, 275-286, 2016.

Three-way Analysis

- J.J. Qi, T. Qian, L. Wei, The connections between three-way and classical concept lattices, Knowledge-Based Systems 91, 143-151, 2016.
- J. Chen, Y.P. Zhang, S. Zhao, Multi-granular mining for boundary regions in three-way decision theory, Knowledge-Based Systems 91, 287-292, 2016.
- M. Li, G.Y. Wang, Approximate concept construction with three-way decisions and attribute reduction in incomplete contexts, Knowledge-Based Systems 91, 165-178, 2016.
- A.V. Savchenko, Fast multi-class recognition of piecewise regular objects based on sequential three-way decisions and granular computing, Knowledge-Based Systems 91, 252-262, 2016.
- J. Qi, L. Wei, Y.Y. Yao, Three-way formal concept analysis, RSKT 2014, LNCS vol. 8818, pp. 732-741, 2014.

Decision-Support Systems

- J.T. Yao, N. Azam, Web-based medical decision support systems for three-way medical decision making with game-theoretic rough sets, IEEE Transactions on Fuzzy Systems 23 (2014) 3-15.
- N. Azam, Investigating Decision making with Game-theoretic Rough Sets, PhD thesis, University of Regina, 2014

Sentiment Analysis

- Z. Zhou, W.B. Zhao, L. Shang, Sentiment analysis with automatically constructed lexicon and three-way decision, RSKT 2014, 777-788, 2014.
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Abnormality Detection

- W. Li, Z. Huang, Q Li, Three-way decisions based software defect prediction, Knowledge-Based Systems 91, 263-274, 2016.
- L.B. Zhang, H.X. Li, X.Z. Zhou, Cost-sensitive sequential three-way decision for face recognition, RSEISP 2014, 375-383, 2014.
- Y.L. Liu, L. Pan, X.Y. Jia, Three-way decision based overlapping community detection, RSKT 2013, 279-290, 2013.

Three-way Approximations of Fuzzy Sets

- X.F. Deng, Y.Y. Yao, Decision-theoretic three-way approximations of fuzzy sets, Information Sciences 279 (2014) 702-715.

Concluding Remarks

- We can study rough set theory at different levels of depth and detail.
- We can study rough set theory from different perspectives.
- It is important to pay attention to semantical issues of rough set theory:
  - Increase the chance of proper applications and meaningful generalizations.
  - Avoid misuses of a theory and meaningless generalizations.

Thank you!

- Welcome again to the wonderful world of rough sets.
- Best wishes to your journey.

- For more information, see
  `http://www.cs.uregina.ca/~yyao`

Appendix: Examples

As examples to illustrate studies on a) contents, b) methodology, and c) scientometrics, this appendix contains my annotated sample publications on a) and b), and publications by Dr. J.T. Yao's research group and Dr. Z. Suraj's research group.

Rough Sets and Generalized Rough Sets: My annotated sample publications

Yiyu Yao (姚一豫)

October 5, 2016

Three main issues in a study of rough sets

- Contents:
  - What is rough set theory?
  - How to interpret rough set theory?
  - What can I do?
- Methodology:
  - How to study rough set theory?
  - How can I do?
- Who's Who:
  - What are the major schools of thought?
  - What are the main milestones?
  - Who can I follow?

Contents

- Basic results
  - Two views of rough sets
  - Three definitions of rough set approximations
  - Two representations of rough set approximations
  - Two forms for defining rough set approximations
- Generalized rough set models
  - Generalizations based on different classes of binary relations.
  - Generalizations based on coverings.
  - Generalizations based other mathematical structures, such as Boolean algebras, topological spaces, lattices, poset, etc.
  - Probabilistic rough sets:
    - Decision-theoretic rough sets.
    - Information-theoretic rough sets.
    - Confirmation-theoretic rough sets.
  - Three-way decisions with rough sets.

Two views of rough sets

- Set-oriented views: a rough set is a family of sets, a rough set is a special kind of fuzzy set (i.e., rough membership function)
- Operator-oriented views: lower and upper approximations are unary operators on the power set of the universe (i.e., set theory with added operators)
- Yao, Y.Y., Two views of the theory of rough sets in finite universes. International Journal of Approximation Reasoning 15 (1996) 291-317.

Three definitions of rough set approximations

- Element based definition: focusing on individual elements of the universe.
- Granule based definition: focusing on equivalence classes.
- Subsystem based definition: focusing on an atomic Boolean algebra with the set of equivalence classes as its atoms.
- Yao, Y.Y., On generalizing rough set theory, Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing, Proceedings of the 9th International Conference (RSFDGrC 2003), LNAI 2639, pp. 44-51, 2003.

Two representations of rough set approximations

- Rough set theory in terms of a pair of lower and upper approximations.
- Rough set theory in terms of three regions.
- They define each other.
- One can use any one of them as a primitive notion and treat the other as a derived notion.
- Yao, Y.Y., Rough sets and three-way decisions, in: Ciucci, D., Wang, G.Y., Mitra, S., Wu, W.Z. (eds.) RSKT 2015. LNCS (LNAI), vol. 9436, pp. 59-70, (2015).

Two forms for defining rough set approximations

- Rough set approximations can be defined based on set inclusion and set intersection:

$$\begin{aligned} \underline{apr}(A) &= \{x \mid x \in U, [x]_E \subseteq A\} \\ \overline{apr}(A) &= \{x \mid x \in U, [x]_E \cap A \neq \emptyset\}. \end{aligned}$$

- Rough set approximations can be defined based only on set-inclusion:

$$\begin{aligned} \underline{apr}(A) &= \{x \mid x \in U, [x]_E \subseteq A\} \\ \overline{apr}(A) &= \{x \mid x \in U, \neg([x]_E \subseteq A^c)\}. \end{aligned}$$

- Yao, Y.Y. and Deng, X.F., Quantitative Rough Sets based on Subsethood Measures, Information Sciences 267, 306-322, 2014.

Generalizations based on different classes of binary relations.

- Element based definition:

$$\begin{aligned}
\underline{apr}(A) &= \{x \mid x \in U, [x]_E \subseteq A\} \\
&= \{x \mid x \in U, \forall y \in U[xEy \Longrightarrow y \in A]\}, \\
\overline{apr}(A) &= \{x \mid x \in U, [x]_E \cap A \neq \emptyset\} \\
&= \{x \mid x \in U, \exists y \in U[xEy, y \in A]\};
\end{aligned}$$

- Establishes a connection to modal logic
- Leads to generalized rough sets using arbitrary binary relations
- Yao, Y.Y. and Lin, T.Y., Generalization of rough sets using modal logic, Intelligent Automation and Soft Computing, an International Journal 2 (1996) 103-120.
- Yao, Y.Y., Relational interpretations of neighborhood operators and rough set approximation operators. Information Sciences 101 (1998) 239-259.

Generalizations based on coverings.

- Granule based definition:

$$\begin{aligned}
\underline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in U/E, [x]_E \subseteq A\} \\
&= \bigcup\{X \mid X \in U/E, X \subseteq A\}, \\
\overline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in U/E, [x]_E \cap A \neq \emptyset\} \\
&= \bigcup\{X \mid X \in U/E, X \cap A \neq \emptyset\};
\end{aligned}$$

- Establishes a connection to granular computing
- Leads to generalized rough sets by using coverings
- Yao, Y.Y., Relational interpretations of neighborhood operators and rough set approximation operators. Information Sciences 101 (1998) 239-259.
- Yao, Y.Y. and Yao, B.X., Covering based rough set approximations, Information Sciences, Vol. 200, 91-107, 2012.

Generalizations based on other mathematical structures

- Subsystem based definition:

$$\begin{aligned} \underline{apr}(A) &= \bigcup\{X \mid X \in \mathbb{B}(U/E), X \subseteq A\}, \\ \overline{apr}(A) &= \bigcap\{X \mid X \in \mathbb{B}(U/E), A \subseteq X\}. \end{aligned}$$

- Establishes a connection to other mathematical structures, including topological space, closure systems, etc.
- Leads to generalized rough sets using other structures
- Yao, Y.Y., On generalizing Pawlak approximation operators. In: RSCTC 1998, LNCS (LNAI) 1424, pp. 298-307.

Decision-theoretic rough sets (DTRS)

- Introduce a general definition of probabilistic rough set approximations defined by a pair of thresholds on conditional probability.
- Have a solid theoretic foundation on Bayesian decision theory.
- Compute the pair of threshold based on Bayesian decision theory.
- Cover majority of probabilistic rough set model that were introduced either before or after DTRS.

Decision-theoretic rough sets (DTRS)

- Yao, Y.Y., Probabilistic rough set approximations, International Journal of Approximate Reasoning 49(2): 255-271, 2008.
- Yao, Y.Y., Probabilistic approaches to rough sets, Expert Systems 20 (2003) 287-297.
- Yao, Y.Y., Decision-theoretic rough set models, RSKT 2007, LNCS (LNAI), vol. 4481, pp. 1-12, 2007.
- Yao, Y.Y. and Wong, S.K.M. A decision theoretic framework for approximating concepts, International Journal of Man-machine Studies 37(6): 793-809, 1992.
- Yao, Y.Y., Wong, S.K.M., and Lingras, P., A decision-theoretic rough set model, Methodologies for Intelligent Systems 5, Ras, Z.W., Zemankova., M., and Emrichm M.L. (Eds.), New York: North-Holland, pp. 17-25, 1990.

Information-theoretic rough sets (ITRS)

- Provide an alternative way to determine the pair of threshold based on Shannon information theory.
- Deng, X.F., and Yao, Y.Y., A multifaceted analysis of probabilistic three-way decisions, Fundamenta Informaticae 132: 291-313, 2014.

Confirmation-theoretic rough sets (CTRS)

- Provide an alternative probabilistic methods for rough sets for the purpose of feature selection.
- Have a solid foundation based on Bayesian confirmation theory.
- Yao, Y.Y. and Zhou, B., Two Bayesian approaches to rough sets, European Journal of Operational Research 251, 904-917, 2016.

Three-way Decisions with Rough Sets

- Interpretation of three-way decisions:
  - Positive region: acceptance, rules of acceptance
  - Negative region: rejection, rules of rejection
  - Boundary region: noncommitment, further information/evidence is required.

Three-way Decisions

- Y.Y. Yao, Three-way decision: An interpretation of rules in rough set theory. In: RSKT 2009, 2009, LNCS (LNAI), vol. 5589, pp. 642-649 (2009).
- Y.Y. Yao, Three-way decisions with probabilistic rough sets. Information Sciences 180 (2010) 341 353.
- Y.Y. Yao, The superiority of three-way decisions in probabilistic rough set models. Information Sciences 181 (2011) 1080-1096.
- Y.Y. Yao, An outline of a theory of three-way decisions. In: J.T. Yao, Y. Yang, R. Slowinski, S. Greco, H.X. Li, S. Mitra, L. Polkowski (eds.) RSCTC 2012, LNCS (LNAI), vol. 7413, pp. 1-17 (2012)

Methodology

- Two approaches to studying rough set theory.
- Two sides of rough set theory.
- Duality through an application of the square of oppositions.

Two approaches to study rough set theory

- Constructive methods: lower and upper approximations are constructed based on other notions such as binary relations, partitions, Boolean algebras, lattice, etc.
- Axiomatic methods: lower and upper approximations are treated as unary operators, and axioms are used to characterize the two operators
- Yao, Y.Y., Constructive and algebraic methods of the theory of rough sets. Information Sciences 109 (1998) 21-47.

Two sides of rough set theory

- Conceptual formulation: focusing on "what," increasing understanding of various notions and concepts.
- Computational formulation: focusing on "how," increasing speed of computation.
- Yao, Y.Y. The two sides of the theory of rough sets, Knowledge-based Systems 80, 67-77, 2015.

Duality through an application of the square of oppositions

- Study duality of many notions in rough sets, including relations, approximations, and attribute classification.
- Provide a simple geometric illustration of connections between different notions.
- Yao, Y.Y., Duality in Rough Set Theory Based on the Square of Opposition, Fundamenta Informaticae, Vol. 127, 49-64, 2013.

Who's Who

- Identify important schools of thought.
- Identify main milestones.
- Identify whom to follow.
- Papers by **other** researchers:
  - J.T. Yao, Y. Zhang, A Scientometrics Study of Rough Sets in Three Decades, in P. Lingras et al. (Eds.): RSKT 2013, LNCS(LNAI) 8171, pp28-40, 2013.
  - Suraj, Z., Grochowalski, P., Lew, Ł.: Discovering patterns of collaboration in rough set research: Statistical and graph-theoretical approach. In: Yao, J., Ramanna, S., Wang, G., Suraj, Z. (eds.) RSKT 2011. LNCS, vol. 6954, pp. 238–247. Springer, Heidelberg (2011)

Thank you for your attention

Thanks!

谢谢!

For more information, see
http://www.cs.uregina.ca/∼yyao